

Thesis

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Combination synchronization of some non-integer order chaotic systems

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DEDICATION

Praise be to God, who in our hearts is the love of knowledge, the passion for learning, and the aspiration for the highest and most honorable ranks. Praise be to God, who has occupied us with knowledge alone. Praise be to God, who has made us among the elite of society. Praise be to God, who has guided us, aided us, and facilitated our path. This path is full of challenges and obstacles, but with God's grace and favor, we have reached this goal. Praise be to God, who has been our support, our backer, and our guide, especially in moments of despair. He is kind to us when fate throws us into labyrinths, and He illuminates our paths after we have almost lost hope in ourselves. He who does not thank people does not thank God. Let us not forget to thank those who have supported us, even if only a little, throughout our academic journey. We mention, but are not limited to: To those who have been my support and my backer after God; to those who have always remembered me and remember me in their prayers; who have been like a mountain, supporting me in moments of despair, failure, and weakness, and who understand what is on my mind without me even speaking; to those who have supported me with all their strength. All thanks to you, my dear mother. May God prolong your life and reward you with all the best on my behalf. To my dear father, who instilled in us a love of knowledge and learning, supported us, and was keen to educate us, I extend my sincere thanks and gratitude. To my brothers, Abdel Nasser, Abdel Salam, and Hamza, who were good role models for me and always aspired to the best. I especially want to mention my brother Abdel Salam, who always supported me, especially when I was taking the doctoral exam, tirelessly. He believed in me and appreciated my ability to succeed at a time when I had lost hope and self-confidence. To my dear sisters, you are my constant support and have been behind me every step

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ABSTRACT

The primary goal of this thesis is to examine the synchronization of the combination of specific fractional-order chaotic systems. Two main contributions of this thesis which are:

1. The first contribution is study compound synchronization between three unified chaotic systems with classical order of derivatives and one unified chaotic system of fractional order is presented.

2. The second contribution presents a novel approach to examine the issue of chaos combination anti-synchronization amongst three chaotic systems with fractional-order impacted by random noise.

The study of the proposed synchronization schemes relies on active control and adaptive control techniques to analyse the convergence and stability of the error of synchronization systems. The proposed synchronization schemes are illustrated using some examples and numerical simulations.

Keywords: Fractional chaotic systems, Combination synchronization, Active control, Adaptive control, Random noise.

RÉSUMÉ

Cette thèse a pour objectif principal d'étudier la synchronisation combinée de quelques systèmes chaotiques d'ordre fractionnaire. Elle présente deux contributions importants:

1. La première contribution est pour développer le problème de la synchronisation combinée entre trois systèmes chaotiques unifiés d'ordre de dérivation classique et un système chaotique unifié d'ordre fractionnaire.
2. La deuxième contribution est de présenter une nouvelle approche pour étudier le problème de l'anti-synchronisation combinée entre trois systèmes chaotiques d'ordre fractionnaire perturbés par un bruit aléatoire.

L'étude des schémas de synchronisation proposés s'appuie à la fois sur des techniques de contrôle continu et de contrôle adaptatif pour analyser la stabilité et la convergence des systèmes des erreurs de synchronisations. Les schémas de synchronisation proposés sont illustrés par des exemples et des simulations numériques.

Mots-clés: Synchronisation combinée, Systèmes chaotiques fractionnaires, Contrôle adaptatif, Contrôle continu, Bruit aléatoire.

ملخص

تهدف هذه الأطروحة بشكل أساسي إلى دراسة التزامن المركب لبعض الأنظمة الفوضوية ذوات رتب اشتقاق كسرية. المساهمات الرئيسية لهذه الأطروحة هي كما يلي:

1. تقديم مشكلة التزامن المركب بين ثلاثة أنظمة فوضوية موحدة ذوات رتب اشتقاق كلاسيكي ونظام فوضوي موحد ذو رتبة اشتقاق كسرية .
 2. المساهمة الثانية تتمثل في تقديم نهج جديد لدراسة مشكلة التزامن المضاد المشترك بين ثلاثة أنظمة فوضوية ذوات رتب اشتقاق كسرية مضطربة بالضوضاء العشوائية. تعتمد دراسة مخططات المزامنة المقترحة على كل من تقنيات التحكم المستمر والتحكم التكيفي لتحليل استقرار وتقارب أنظمة خطأ المزامنة. يتم توضيح مخططات المزامنة المقترحة من خلال الأمثلة والمحاكاة الرقمية.
- الكلمات المفتاحية: الأنظمة الفوضوية ذوات رتب اشتقاق كسرية ، المزامنة المركبة، التحكم المستمر، التحكم التكيفي، الضوضاء العشوائية.

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General Introduction

The topic of fractional calculus has attracted a lot of attention in recent years across various areas of science and technology, like viscoelasticity, diffusion modeling, control processing, signal transmission, and so forth [1, 2, 3, 4, 5, 6].

The study of systems' stability with fractional order is more delicate than for their counterparts, integer-order systems. Indeed, based on the theory of the stability of integer-order linear systems, a system is considered stable if and only if the root of the characteristic polynomial has negative real parts, that is to say, is located on the complex plane's left half. Moreover, the notion of linear systems stability with fractional-order is a little different from that the classical systems. It has been noted that a stable fractional-order systems can indeed have a characteristic polynomial roots in the complex plane's right half, which shows that fractional systems are memory systems that are more stable (when the fractional order is less than 1) compared to integer-order systems, and consequently, they display a much more sophisticated dynamic behavior, which is of great importance, in particular in the area of secure communication [7, 8].

The phenomenon of chaos synchronization has garnered significant interest in the study of chaotic dynamical systems as a result of its potential applications across various branches of engineering and information science, including cryptology [9], secure communication [7], image encryption [10], and control processing [4].

The fundamental configuration of a synchronization system involves two systems: one is referred to as the drive system, and the other as the response (reaction) system. It is recalled that these systems may be totally distinct or identical with differing initial conditions. One or more coupling signals are used by the transmitter system to synchronize the reception system. Several synchronization methods to guarantee

the asymptotic stability of synchronization errors [11, 12, 13, 14, 15]. Specific kinds of chaotic systems with fractional order have been proposed. However, all previous methods focus primarily on traditional synchronization of two chaotic systems, which represents a very specific case of combination synchronization. Actually, the concept of combination synchronization of one response system and two drive systems was first introduced by Luo et al. [16]. This type of synchronization can significantly improve the anti-attack and anti-decoding capabilities in secure information systems. Several approaches to achieving this synchronization in typical chaotic systems of fractional order have been discussed in [17, 18, 19, 20, 21, 22].

Motivated by the above considerations, the work presented in this thesis generally falls within this particular context. It fundamentally concerns the design of the combination synchronization of fractional drive-response chaotic systems. This thesis's main contributions are:

1. The compound synchronization problem of three classical derivation order unified chaotic systems and one unified chaotic system of fractional order is proposed [23].
2. The second contribution introduces a novel approach to studying the issue of chaos combination anti-synchronization among three chaotic systems of fractional order affected by random noise perturbations. [24].

The thesis is structured as:

In Chapter 1, some definitions and fundamental concepts related to dynamical chaotic systems are provided.

In Chapter 2, we provide some general concepts and auxiliary results related to the fractional calculus.

The Chapter 3 is for fractional-order chaotic dynamical systems.

In Chapter 4, we provide definitions related to traditional and combination synchronization.

In Chapter 5, we study the issue of compound combination synchronization of three integer unified chaotic systems and one fractional unified chaotic system using fractional control.

In Chapter 6, a new straightforward adaptive control scheme is introduced to ensure robust chaos combination anti-synchronization of three fractional-order chaotic

systems with completely boundary noise and unidentified parameters.

Finally, in conclusions and perspectives, we encapsulate the contributions made within the thesis and we propose new avenues for future study, along with suggestions for improvements and perspectives in the considered field.

Classical dynamical systems and chaos

A dynamic system generally refers to phenomena that change and develop over time. The term "system" refers to a set of state variables that vary with time, along with the interactions that connect them. These state variables form a mathematical space known as the "phase space." Such systems can be represented using differential equations or mathematical models, and they are used to analyze and predict the behavior of various phenomena in fields such as physics, economics, engineering, and biology.

Chaos is commonly characterized as a specific dynamical system behavior that includes sensitivity to beginning conditions, nonlinearity, determinism, unpredictability, and irregularity.

In this chapter, the main concepts about dynamical systems and chaos are covered, including some definitions about continuous or discrete dynamical systems like fixed point, attractor, invariant set, trajectory, limit set, flows, theorem of existence and uniqueness, bifurcation, stability of non-linear systems, and some characterization of chaotic behavior.

1.1 Dynamical systems

In the mathematical context, a dynamical system is a structure that changes over time.

Dynamical systems are classified into two types: continuous or discrete dynamical systems.

1.1.1 Continuous dynamical systems

A dynamical system that is continuous can be defined by a differential equations system as follows:

$$\dot{x} = f(x, t), \quad (1.1)$$

the function f describes how the state vector x evolves over time t .

Example 1.1.1. *As an example of continuous dynamical system, the simple pendulum model (without damping), which can be described by:*

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -\frac{g}{l} \sin(x(t)), \end{cases}$$

with:

- * x, y represent the state vectors.
- * x defines the pendulum's angle.
- * y denotes the rate of change of the pendulum's angle.
- * g denotes the acceleration caused by gravity.
- * l represents the pendulum's length .

Example 1.1.2. *The model of Lotka–Volterra predator–prey is:*

$$\begin{cases} \dot{x}_1(t) = \alpha x_1 - \beta x_1 x_2, \\ \dot{x}_2(t) = -\gamma x_2 + \delta x_1 x_2, \end{cases}$$

with:

- * The number of prey at time t is represented by x_1 .
- * The population of predator at the time t is represented by x_2 .
- * The parameter α describes the natural increase in the prey population x_1 .
- * δ is the rate at which the predator x_2 increase by consuming the prey x_1 .
- * γ represents the predator x_2 's natural death rate.

1.1.2 Discrete dynamical systems

A discrete dynamical system is described by iterative functions as follows :

$$x_{k+1} = f(x_k), \quad (1.2)$$

where f is any function that characterizes the dynamics of the system and the variable x_k denotes the state of system (1.2) at time step k .

Example 1.1.3. A famous example of discrete dynamical system is the logistic growth model, which describes population dynamics:

$$x_{k+1} = \lambda x_k \left(1 - \frac{x_k}{K}\right). \quad (1.3)$$

Where:

- * x_k is the population at the time step k .
- * x_{k+1} is the population at the next time step $k + 1$.
- * λ is the growth rate.
- * K is the carrying capacity.

Example 1.1.4. The model of Bacteria Growth, is given by:

$$N_{t+1} = rN_t.$$

Where:

- * N_t is the number of bacteria at time step t .
- * r is the growth rate .

This model uses a discrete-time exponential growth model to simulate the growth of bacteria in a controlled environment . Suppose that we start with 100 bacteria ($N_0 = 100$) and they double every hour ($r = 2$).

Time Step (t)	Bacteria Count (N_t)
0 (initial)	100
1	$100 \times 2 = 200$
2	$200 \times 2 = 400$
3	$400 \times 2 = 800$
4	$800 \times 2 = 1600$

Example 1.1.5. The Standard-Taylor Map, also known as the Chirikov-Taylor Map, is a famous two-dimensional discrete dynamical system that exhibits chaotic behavior. It's commonly used in the study of dynamical systems, especially in chaos theory. The equations for the Standard-

Taylor Map are:

$$\begin{cases} \theta_{n+1} = \theta_n + p_{n+1}(\text{mod}2\pi), \\ p_{n+1} = p_n + K \sin(\theta_n)(\text{mod}2\pi). \end{cases}$$

With:

- $\theta_n \in [0, 2\pi)$ is the angular variable (or position),
- $p_n \in \mathbb{R}$ is the momentum variable (or velocity),
- K is the nonlinearity parameter that controls the degree of chaos in the system,
- n is the discrete time step index.

Remark 1.1.1. The system (1.1) or (1.2) is called **autonomous** if its future behavior depends only on its current state, with no change over time. Otherwise, if the function f clearly is dependent on time, it's called **non-autonomous**.

In the rest of this work, we will be interested in continuous autonomous systems. Now, let's consider a system with first-order differential equation:

$$\dot{x}(t) = f(x), \tag{1.4}$$

with $x = (x_1, x_2, \dots, x_n)^T$, $f = (f_1, f_2, \dots, f_n)^T$.

1.2 Cauchy-Lipschitz theorem

The following theorem ensures that under conditions, a differential system (1.5) has unique solution

Theorem 1.2.1.

Let the differential system:

$$\begin{cases} \dot{x}(t) = f(x), \\ x(0) = x_0, \end{cases} \tag{1.5}$$

with f is a function defined in a neighbourhood of the initial condition.

Assume that:

* f is continuous function with respect to the variable x .

* f is a function that is Lipschitz and continuous with respect to x ,

then the problem has unique solution in the neighbourhood of the initial condition.

1.3 Some concepts about continuous dynamical systems

In the current section, we give some important notions and their definitions about dynamical systems that are continuous, including fixed points, trajectories, flows, and invariant sets. We also talk about attractors and their types. We move after that to periodic orbits and the technique to investigate their stability, which is Poincaré-Bendixon, vector field divergence, and Poincaré's application.

1.3.1 Flow

A flow of a dynamic system (1.4) is an application that describes how a point in the phase space evolves over time. It can be defined by:

$$\phi_t(x_0) = x(t),$$

where:

- * $x(t)$ denotes the system's state at the time t .
- * ϕ_t represents the function of flow .
- * x_0 represents the initial condition at time $t = 0$, which is crucial because it determines the specific path that the system will follow through its phase space over time.

The flow function ϕ_t has these characteristics:

- (i) ϕ_t is a continuous function.
- (ii) $\phi_0(x_0) = x_0$.
- (iii) $\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0))$.

Example 1.3.1. *Let's assume the system:*

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = -x_1 \end{cases} \quad (1.6)$$

Figure 1.1 shows the flow of the system (1.6) .

1.3.2 Trajectory

A trajectory is defined by a sequence of states generated by the flow function ϕ_t from an initial condition x_0 , i.e., if $x(t) = \phi_t(x_0)$, then a trajectory is the set of the point $x(t)$,

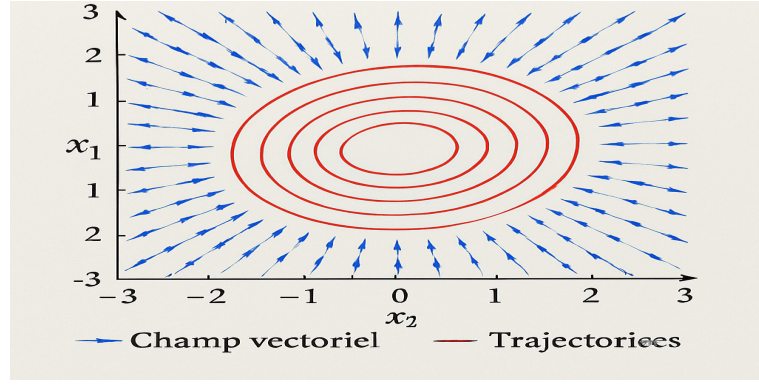


Figure 1.1: Flow of the linear continuous system (1.6)

for all $t \geq 0$. In each trajectory corresponding to an initial condition, we can see how two different starting points can lead to two distinct paths within the same dynamical system.

1.3.3 Fixed point

A fixed point (critical point, stationary point, equilibrium point) represents a state where the variables of the system don't change over time; this indicates that the derivative of each state of the system is equal to zero. That is:

$$\dot{x}(t) = 0 \implies f(x) = 0. \quad (1.7)$$

For nonlinear dynamical systems stability, a fixed point is essential as it determines the long-term behavior of the system in response to external forces.

1.3.4 α -limit set and ω -limit set

The α -limit set and ω -limit set are concepts employed in analyzing the behavior of differential equation solutions as time progresses to negative or positive infinity.

- The α -limit is the set of points where the trajectory of the system approaches as time moves backward. It is defined as:

$$\alpha(x_0) = \left\{ y \in \mathbb{R}^n : \exists t_k \longrightarrow -\infty, \text{ such that } \phi(t_k, x_0) \rightarrow y \text{ as } k \rightarrow +\infty \right\}.$$

- The ω -limit is the set of points such that the trajectory of the system approaches as time t goes to $+\infty$. It is defined as:

$$\omega(x_0) = \left\{ y \in \mathbb{R}^n : \exists t_k \longrightarrow +\infty, \text{ such that } \phi(t_k, x_0) \rightarrow y \text{ as } k \rightarrow +\infty \right\},$$

where $\phi(t_k, x_0)$ is the solution of a dynamical system at time t starting from the initial starting condition x_0 .

Example 1.3.2. Consider the system in $\mathbb{R}^2$:

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -y.$$

This system has the solution:

$$x(t) = x_0 e^{-t}, \quad y(t) = y_0 e^{-t}.$$

1. As $t \rightarrow +\infty$, we see that $(x(t), y(t)) \rightarrow (0, 0)$. So, the ω -limit set of any non-zero initial point is $\{(0, 0)\}$.
2. As $t \rightarrow -\infty$, the trajectory diverges to infinity, meaning the α -limit set of any non-zero initial point is empty ($\alpha(x, y) = \emptyset$).

1.3.5 Periodic orbits (Cycles)

The following definitions introduce several concepts related to the periodic orbits (cycles).

Definition 1.3.1. A periodic orbit (cycle) is any closed trajectory of the system (1.4), that is not a fixed point.

Definition 1.3.2. Let γ be a periodic orbit of (1.4). γ is said to be **stable**, if $\forall \epsilon > 0, \exists V$ a neighborhood of the orbit γ , such that:

$$\forall x \in V \text{ and } t > 0, d(\phi(t, x), \gamma) < \epsilon.$$

Definition 1.3.3. A periodic orbit γ is considered **asymptotically stable** when the following two conditions are satisfied:

- 1- γ is stable.
- 2- $\forall x \in V$ (a neighborhood of γ), we have:

$$\lim_{t \rightarrow +\infty} d(\phi(t, x), \gamma) = 0.$$

1.3.6 Invariant set

An **invariant set** E under the flow ϕ_t is a region in phase space, where once a trajectory enters, it cannot leave for all future times. Formally, it can be represented as:

$$\forall x \in E, \phi_t(x) \in E, \forall t \geq 0.$$

1.3.7 Attractor

In dynamical system, an **attractor** is an invariant set A , such that, for initial condition x_0 in neighborhood of A , all trajectories of the system, $\phi(t, x_0)$, $t \geq 0$, will approach A , as time t tends to infinity.

1.3.8 Type of attractors

They are classified as "strange (chaotic) attractors" and "regular attractors."

•**Regular attractors:** There are three different types of regular attractor:

- * Point attractor (fixed point): It is the simple attractor, where the trajectories of the system converge to a single point.
- * Limit cycle attractor: it is a periodic orbit, where the trajectories of the system converge to a closed loop.
- * Quasi-periodic attractor (Torus):

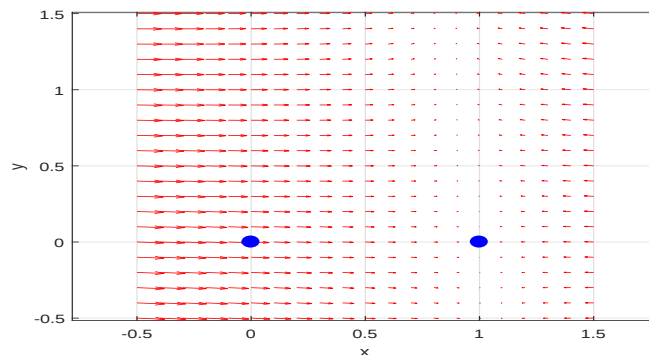


Figure 1.2: Fixed point.

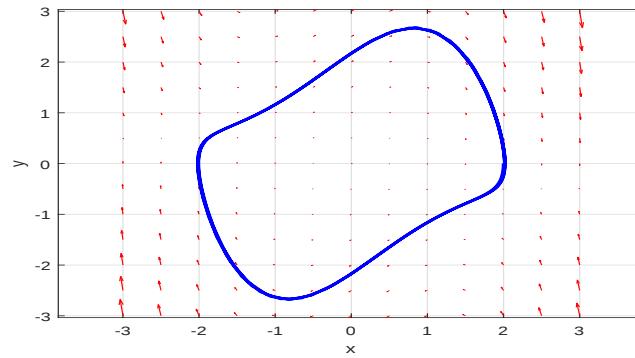


Figure 1.3: Limit cycle attractor

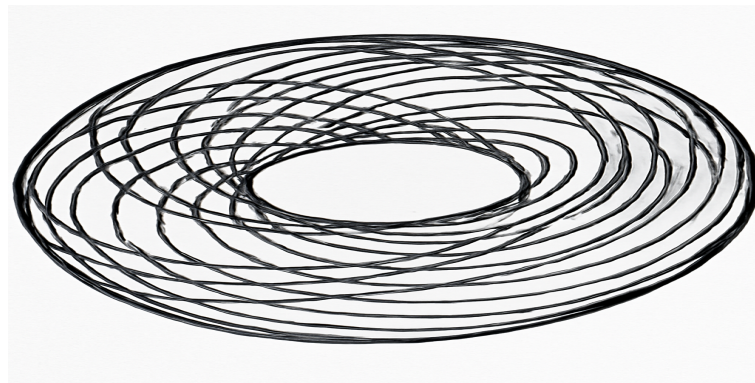


Figure 1.4: Quasi-periodic attractor (Torus).

• **Strange attractor:** It found it in chaotic or hyperchaotic systems, where the trajectories of the system exhibit a complex fractal-like behavior.

This type is characterized by:

- * Their volume is zero.
- * Their dimension is fractal not an integer.
- * Their trajectories have complex behavior.
- * Sensitivity to initial conditions.

Figure 1.5 represents one of the most famous strange attractors, which is the attractor of the Lorenz system,

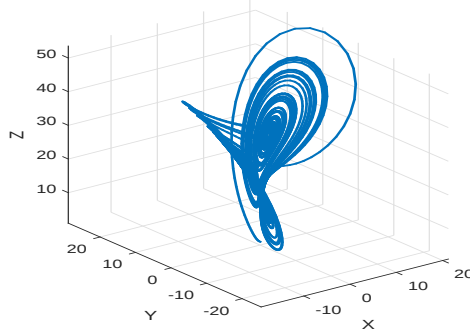


Figure 1.5: Lorenz attractor.

1.3.9 Poincaré-Bendixson Theorem

Theorem 1.3.1. *Let D be a plane's bounded attracting domain, then each trajectory in D has the following β -limit:*

1. *Either a fixed point.*
2. *Or a periodic orbit.*
3. *Or the union of equilibrium sites with regular orbits that link them (either homoclinic or heteroclinic orbits).*

Example 1.3.3. *Our purpose is to establish the existence of a periodic orbit using the Theorem of Poincaré-Bendixson*

Let us examine the following system:

$$\begin{aligned}\dot{x} &= y + \frac{1}{4}x(1 - 2r^2), \\ \dot{y} &= -x + \frac{1}{2}y(1 - r^2),\end{aligned}$$

with: $r^2 = x^2 + y^2$.

Let's first look for the fixed points of this vector field. The first equation gives us:

$$y = -\frac{1}{4}x(1 - r^2).$$

After substituting into the second equation, we find:

$$-x \left(1 + \frac{1}{8}(1 - 2r^2))(1 - r^2) \right) = 0.$$

So, a fixed point exists at the origin $(x, y) = (0, 0)$, and any other fixed point must satisfy the equation:

$$2r^4 - 3r^2 + 9 = 0.$$

Since the solutions of previous equation are not real, the origin is unique fixed point of the system.

Using the polar coordinates, we get:

$$\dot{r} = \frac{1}{4}r(1 + \sin^2 \theta^2) - \frac{1}{2}r^3.$$

For $r \leq \sqrt{\left(\frac{1 + \sin^2(\theta)}{2}\right)}$, we have $\dot{r} > 0$, for $r \geq \sqrt{\left(\frac{1 + \sin^2(\theta)}{2}\right)}$, we have $\dot{r} < 0$.

Then

$$0 \leq \sin^2 \theta \leq 1,$$

$$1 \leq 1 + \sin^2 \theta \leq 2,$$

$$\sqrt{\frac{1}{2}} \leq \sqrt{\frac{1 + \sin^2 \theta}{2}} \leq 1,$$

$$\text{that is, } \frac{\sqrt{2}}{2} \leq r \leq 1.$$

Let:

$$M = \left\{ (r, \theta) \mid \frac{\sqrt{2}}{2} \leq r \leq 1 \right\}.$$

Because the origin represents the only fixed point of the system, the region M contains no equilibria and all trajectories remain in M for all $t \geq 0$ and M is compact, it is therefore a trapping region that contains no equilibria. By the Theorem of Poincaré-Bendixson, there is just one periodic orbit in M at least.

1.3.10 Vector field's divergence

The divergence of a system at a point quantifies the behavior of trajectories in its vicinity; specifically, a negative divergence indicates contraction and convergence of trajectories, while a positive divergence signifies expansion and divergence of trajectories. Here is its definition.

Definition 1.3.4. Let's the two dimension systems :

$$\begin{cases} \dot{x}_1 = k(x_1, x_2), \\ \dot{x}_2 = h(x_1, x_2), \end{cases} \quad (1.8)$$

Let Y be a vector field described by the differential system (1.8) in the phase space.

The following expression gives the divergence of Y at the point (x_1, x_2) :

$$\operatorname{div}(Y)(x_1, x_2) = \frac{\partial k}{\partial x_1}(x_1, x_2) + \frac{\partial h}{\partial x_2}(x_1, x_2).$$

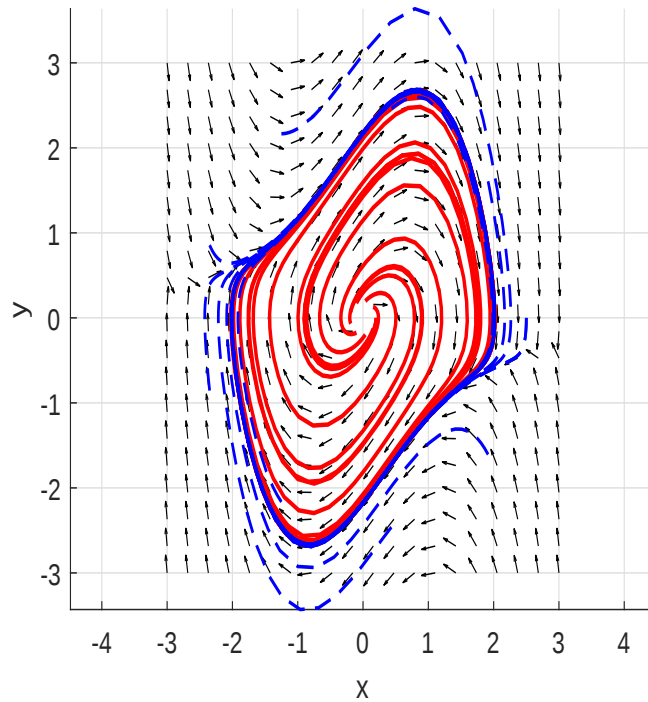


Figure 1.6: Stable limit cycle

Proposition 1.3.1. (Bendixson's negative criterion) Let us consider the dynamical system (1.8) and let E be a simply connected region, which is "all in one piece" or "without holes".

If the number

$$\frac{\partial k}{\partial x_1} + \frac{\partial h}{\partial x_2},$$

has a constant sign on E , there is no limit cycle in E .

Example 1.3.4. Consider the following linear system :

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 = k(x_1, x_2) \\ \dot{x}_2 = cx_1 + dx_2 = h(x_1, x_2), \end{cases} \quad (1.9)$$

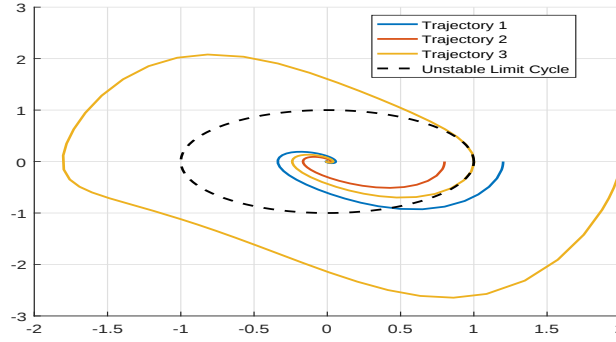


Figure 1.7: Unstable limit cycle

where $a + d \neq 0$.

We have

$$\frac{\partial k}{\partial x_1} + \frac{\partial h}{\partial x_2} = a + d.$$

Since $a + d$ has a constant sign, for each point in the plane, so, according to Bendixson's negative criterion 1.3.1, a linear system (1.9) cannot have a limit cycle.

Proposition 1.3.2. (Dulac's negative Criterion)

Considering the dynamical system (1.8), and let E be a simply connected region, and let $A(x_1, x_2)$ in E be strictly positive, continuous and differentiable function.

Let's consider the quantity:

$$\frac{\partial Ak}{\partial x_1} + \frac{\partial Ah}{\partial x_2}.$$

If this quantity has a constant sign in E , then there is no limit cycle contained in E .

Example 1.3.5. Let us examine the next system::

$$\dot{x} = x(1 - x - ay) = k(x_1, x_2)$$

$$\dot{y} = y(1 - y - bx) = h(x_1, x_2)$$

with a and b are two parameters strictly positive.

Let $A(x, y) = \frac{1}{xy}$ a continuous differentiable and strictly positive the function for all $x > 0, y > 0$, we have:

$$Ak = \frac{1 - x - ay}{y}, \quad Ah = \frac{1 - y - bx}{x}$$

$$\frac{\partial Ak}{\partial x} + \frac{\partial Ah}{\partial y} = -\frac{1}{x} - \frac{1}{y}$$

This quantity is strictly negative at every point within the interior of the positive quadrant. Therefore, by Dulac's criterion, it can be concluded that no limit cycle exists within the positive quadrant.

1.3.11 Poincaré's Application

The Poincare application, also known as the first return map, was created by Henri Poincare in 1881. It is perhaps, the most fundamental technique for examining the stability of periodic orbits. The Poincare application's concept is rather easy [52].

Let us consider the dynamical system (1.8) governed by a flow $\phi_t: E \rightarrow E$, where E is the state space.

Assume that:

1. The system (1.8) exhibits a closed trajectory γ .
2. At x_0 , let Σ (called the Poincaré section) be a lower-dimensionel submanifold in E , which is a hyperplane perpendicular to γ ,

The Poincaré application is every application $P: \Sigma \rightarrow \Sigma$, which is expressed by:

$$P(x) = \phi_\tau(x),$$

with τ is the first return time to the section Σ for the point x . i.e., τ represents the smallest positive time when $\phi_\tau \in \Sigma$.

This application reduces the analysis and the continuous dynamical systems stability to discrete dynamical systems in Σ .

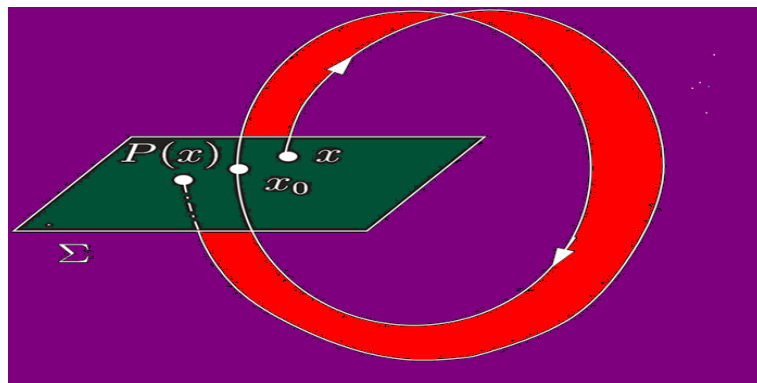


Figure 1.8: Poincaré's section (3.109).

Theorem 1.3.2. (*Stability of a periodic orbit via Poincaré's application*) Consider S as an open subset of $\mathbb{R}^2$. assume that $f \in C^1(S)$ and the system (1.8) have a periodic solution of period T , denoted by $\gamma(t)$. Then, at $x = 0$, the Poincaré application derivative $P(s)$ on the straight line Σ normal to:

$$\Gamma = \{x \in \mathbb{R}^2 / x = \gamma(t) - \gamma(0), 0 \leq t \leq T\},$$

is as follows:

$$P'(0) = e^{\int_0^T \nabla \cdot f(\gamma(t)) dt},$$

with:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right),$$

and

$$f = (f_1, f_2)^T.$$

The next corollary summarizes the earlier theory.

Corollary 1.3.1.

i) The limit cycle $\gamma(t)$ is stable if:

$$\int_0^T \nabla \cdot f(\gamma(t)) dt < 0.$$

ii) The limit cycle $\gamma(t)$ is unstable if:

$$\int_0^T \nabla \cdot f(\gamma(t)) dt > 0.$$

Example 1.3.6. Let the planar system:

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2). \end{cases}$$

This system has a cycle :

$$\gamma(t) = (\cos(t), \sin(t))^T.$$

Using the corollary 1.3.1, we get:

$$\frac{\partial k}{\partial x}(x, y) = 1 - 3x^2 - y^2$$

and

$$\frac{\partial h}{\partial y}(x, y) = 1 - x^2 - 3y^2.$$

So:

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial k}{\partial x}(x, y) + \frac{\partial h}{\partial x}(x, y), \\ &= 2 - 4x^2 - 4y^2,\end{aligned}$$

as a result:

$$\nabla f(\cos t, \sin t) = 2 - 4\cos^2(t) - 4\sin^2(t) = -2.$$

Thus

$$\int_0^T \nabla \cdot f(\gamma(t)) dt = -2T < 0,$$

Finally:

$$P'(0) = e^{-2T} < 1.$$

From this, we conclude that the orbit $\gamma(t)$ is a stable limit cycle.

1.4 Stability of non-linear system

Stability is a study of the behavior of the system solutions in the vicinity of equilibrium points and is a key tool in the qualitative study of a dynamical system's solutions.

Suppose that $x^* \in \mathbb{R}^n$ an equilibrium point, is said to be:

- **stable**, if the trajectories of the dynamical system that start close to the equilibrium point stay close to the equilibrium over the time. Mathematically, this can be expressed as:

$$\forall \epsilon > 0, \exists \delta > 0 \quad / \quad \|x(0) - x^*\| < \delta \implies \|x(t) - x^*\| < \epsilon.$$

- **asymptotically stable**: if it is stable and if any trajectory that start near equilibrium point converge to it, when time t converges to infinity. Mathematically, this can be described as:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \|x(0) - x^*\| < \delta, \lim_{t \rightarrow \infty} \|x(t) - x^*\| = 0,$$

where $\|\cdot\|$ represents the norm in $\mathbb{R}^n$.

- **Unstable** if x^* is not stable.

For non-linear systems, there are two techniques in order to investigate their solution's stability, indirect method and direct method.

1.4.1 Indirect method (Method of linearization)

This method is often employed to analyse the stability of nonlinear system by approximating them like linear systems around a fixed point.

We can summarize the idea of this method as follows:

1. Determine the following dynamical system :

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (1.10)$$

2. Determine the fixed points x^* by resolving the following equation:

$$f(x) = 0. \quad (1.11)$$

3. Linearize the system (1.10) around the point x^* .
4. Expand $f(x)$ into a Taylor series around x^* .
5. Examine the eigenvalues of the Jacobian matrix $J_f(x^*)$ to investigate the stability.

Then, we have :

- The fixed point is **asymptotically stable** if all of the Jacobian matrix's eigenvalues have negative real parts.
- If the Jacobian matrix has a pure imaginary eigenvalues, the fixed point is classified as a **center**.
- If an eigenvalue has a positive real part, the fixed point is **unstable**.
- If $J_f(x^*)$ has real eigenvalues of the identical sign, the fixed point is called a **stable node** if the eigenvalues are negative and an **unstable node** if the eigenvalues are positive.
- If the Jacobian matrix has eigenvalues with positive real parts with other eigenvalues with negative real parts, then the fixed point is a **saddle point**.

Example 1.4.1. Let the nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 - x_2^2, \\ \dot{x}_2 = x_1^2 - x_2, \end{cases}$$

The fixed points are: $(0, 0)$ and $(1, 1)$.

The Jacobienne matrix is :

$$J_f(x_1, x_2) = \begin{pmatrix} 1 & -2x_2 \\ 2x_1 & -1 \end{pmatrix}$$

For $(0, 0)$, we have :

$$J_f(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The matrix $J_f(0, 0)$ have two eigenvalues:

$$\lambda_1 = 1, \lambda_2 = -1.$$

$\lambda_1 > 0$ and $\lambda_2 < 0$, thus, the point $(0, 0)$ is unstable .

For the point $(1, 1)$ we have :

$$J_f(1, 1) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

The matrix $J_f(1, 1)$ has two eigenvalues:

$$\lambda_1 = 1, \lambda_2 = -3.$$

$\lambda_1 > 0$ and $\lambda_2 < 0$ thus, the point $(1, 1)$ is unstable.

In the both cases, the fixed point is stable in one direction and unstable in the other direction, which means that the fixed points $(0, 0)$ and $(1, 1)$ are saddle points. Figure (1.9) represents the temporal evolution of the states x_1 and x_2 with initial condition $(x_1, x_2) = (0.01, 0.01)$.

Definition 1.4.1.

Let D_1 and D_2 be two dynamic differential systems defined on the open sets $S_1 \subset \mathbb{R}^2$ and $S_2 \subset \mathbb{R}^2$, respectively. These systems (or their phase portraits) are considered topologically equivalent, If a homeomorphism H mapping exists from S_1 to S_2 , such that the image of the phase portrait of D_1 under H corresponds to the phase portrait of D_2 .

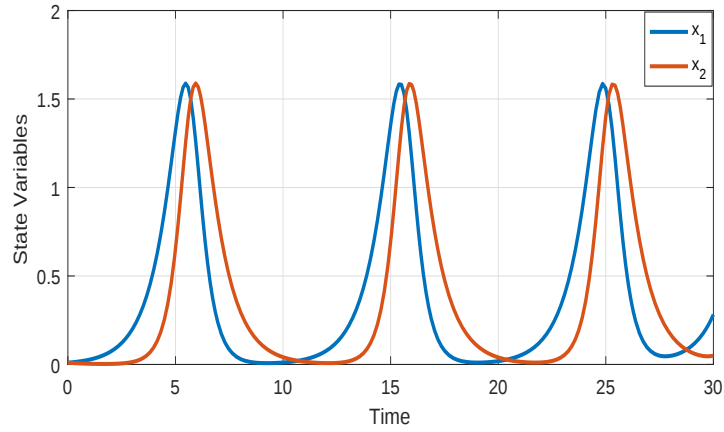


Figure 1.9: Temporal evolution of the states x_1 and x_2 .

Theorem 1.4.1. (Hartman-Grobman Theorem) [53] Consider a nonlinear dynamical system D defined on $S \subset \mathbb{R}^2$ (S is an open set) with a unique equilibrium at $0 \in S$. The phase portrait of the nonlinear system D and that of its linearized system D_1 in the vicinity of the equilibrium are said to be topologically equivalent if the fixed point is hyperbolic (non-zero real part of eigenvalues of the Jacobian matrix at this fixed point) .

1.4.2 Limitation of linearisation

Linearization provides an effective method for studying the local stability of nonlinear dynamical systems by approximating them as linear systems, but care must be taken regarding its limitations. Then, when the method of linearization predicts a focus, a saddle point , or a node, it is valid. However, when the linear system has an eigenvalue with a zero real part, it is not enough to comprehend the dynamics of a nonlinear system. Other techniques must be used in this situation because linearization does not yield definitive information. The following example explains this case:

Example 1.4.2. Let the nonlinear system:

$$\begin{cases} \dot{x} = x_2 - \alpha x_1(x_1^2 + x_2^2), \\ \dot{y} = -x_1 + \alpha x_2(x_1^2 - x_2^2). \end{cases} \quad (1.12)$$

with α is a parameter.

There is one fixed point of the system(1.12) at the origin.

At $(0,0)$, the Jacobian matrix is:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which owns two eigenvalues purely imaginary:

$$\lambda_{1,2} = \pm i.$$

The linearization theorem does not apply in this situation since the origin is a center for all α .

We move to polar coordinates (r, θ) :

$$x_1 = r \cos \theta, x_2 = r \sin \theta, \quad (1.13)$$

then:

$$\begin{cases} r^2 = x_1^2 + x_2^2, \\ \tan \theta = \frac{x_2}{x_1}, \end{cases}$$

which implique

$$\begin{cases} \dot{r} = x_1 \dot{x}_1 + x_2 \dot{x}_2, \\ \frac{\dot{\theta}}{\cos^2 \theta} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2}, \end{cases}$$

By using (1.13), we get:

$$\begin{cases} \dot{r} = \alpha r^3 \\ \dot{\theta} = 1. \end{cases}$$

We note that:

- If $\alpha < 0$, $\dot{r}$ approaches 0; then the trajectories spiral to the origin, indicating that it is asymptotically stable.
- If $\alpha = 0$, the trajectories move around the origin in circles, which is a stable and neutral center.
- Trajectories spiral outward from the unstable origin if $\alpha > 0$.

Thus, the linearization method means that the fixed point is a center, while the original system may have a stable or unstable focus.

The dynamic states of the system (1.14) are shown evolving over time in Figure 1.10, where $\alpha = -1$ and the initial condition is $(x_1, x_2) = (0.1, 0.1)$.

1.4.3 Direct method

A direct method is a powerful way that gives us the information about the stability of a fixed point, not only locally but globally as well. We can rely on it, if the point is not hyperbolic.

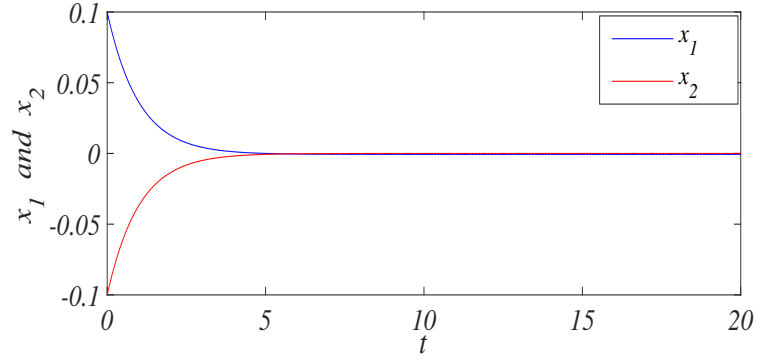


Figure 1.10: Temporel evolution of the states x_1 and x_2 , when $\alpha = -1$.

To begin, we give the definition of a positive function.

Definition 1.4.2. [25] Let V be a function, and S an open set that includes the origin. We say that V is positive definite function (or negative definite function), if it's continuously differentiable and defined on S and satisfies the following characteristics:

1. $V(0,0)=0$.
2. for all $(x, y) \in S - (0, 0)$, $V(x, y) > 0$ (or $V(x, y) < 0$).

Lyapunov function

Let x^* be a system fixed point (1.4), S a neighborhood of x^* and the function $V : S \rightarrow \mathbb{R}$, such that :

$$\begin{cases} V(x^*) = 0, \\ V(x) > 0 \quad \text{if } x \neq x^*, \end{cases}$$

we put

$$\dot{V} = \sum_{k=1}^m \frac{\partial V}{\partial x_k} \dot{x}_k.$$

Then, we have:

Theorem 1.4.2. [25] Suppose that there exist a positive Lyapunov function V .

- (i) If the function $\dot{V} \leq 0$ in $S - \{x^*\}$, x^* is considered as a stable point.
- (ii) If the function $\dot{V} < 0$ in $S - \{x^*\}$, x^* is considered as an asymptotically stable point.
- (iii) If the function $\dot{V} > 0$ in $S - \{x^*\}$, x^* is an unstable point.

Example 1.4.3. Consider the system:

$$\begin{cases} \dot{x} = -y + \beta x(x^2 + y^2), \\ \dot{y} = x + \beta y(x^2 + y^2), \end{cases} \quad (1.14)$$

with β is a parameter.

It evident that $(0,0)$ is the unique equilibrium point of (1.14). The Jacobienne matrix in $(0,0)$ is :

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the eigenvalues are:

$$\lambda_{1,2} = \pm i.$$

This corresponds to a center. However, the linearization Theorem cannot be applied in this case. For solving this problem, we can use the direct technique.

Let the function V :

$$V(x, y) = \frac{1}{2}(x^2 + y^2).$$

Since:

$$\begin{cases} V(0,0) = 0, \\ V(x, y) > 0 \quad \text{if } (x, y) \neq (0,0), \end{cases}$$

then, $V(x, y)$ is positive definite function.

$$\dot{V}(x, y) = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} = \beta(x^2 + y^2)^2 < 0.$$

Thus, the origin is :

1. Asymptotically stable point, for $\beta < 0$.
2. Unstable point for $\beta > 0$.

1.5 Bifurcation

In theory of dynamical system, and particularly in chaos theory, bifurcation refers to a point at which a small modification in the parameters of system leads to a sudden qualitative change in its behavior. In this part, we present at first some types of bifurcation in one dimension includes bifurcation of saddle-node, bifurcation of pitchfork

and transcritical bifurcation, every time we give a diagram for more illustration , after that the bifurcation in dimension greater than one is presented with its types.

1.5.1 Several kinds of bifurcations in codimension 1

Let's consider the non-linear system:

$$\dot{y} = f(y, u), \quad (1.15)$$

with $y \in \mathbb{R}$ is the system solution and u is a parameter .

Applying the Taylor expansion of the function f in the vicinity of the equilibrium point $y^* = 0$, we get :

$$f(y, \mu) = k_{00} + k_{10}\mu + k_{01}y + k_{20}\mu^2 + k_{11}y\mu + k_{02}y^2 + \dots$$

1. Bifurcations of saddle-node

when $k_{10} = 1$, $k_{02} = -1$ and $k_{ij} = 0$ for $(i, j) \neq ((1, 0), (i, j) \neq (0, 2))$, then the system (1.15) become:

$$\dot{y} = \mu - y^2, \mu \in \mathbb{R}.$$

We have three possibility:

- (a) if $\mu < 0$, $f(y, \mu) = \mu - y^2 \neq 0$, then there's no equilibrium point and $\dot{y} < 0 \quad \forall \quad y$.
- (b) if $\mu = 0$, $f(y, \mu) = -y^2 = 0 \rightarrow y = 0$, then $(0, 0)$ is a unique equilibrium point and $\dot{y} < 0 \quad \forall \quad y$.
- (c) if $\mu > 0$, $f(y, \mu) = \mu - y^2 = 0 \rightarrow y = \pm\mu$, then there are two equilibrium points, which are:

$$y_1^* = \sqrt{\mu}, y_2^* = -\sqrt{\mu}$$

then:

$$\begin{aligned} \frac{\partial f}{\partial y_1^*} &= -2\sqrt{\mu} < 0 \rightarrow y_1^* \text{ is stable,} \\ \frac{\partial f}{\partial y_2^*} &= 2\sqrt{\mu} < 0 \rightarrow y_2^* \text{ is unstable.} \end{aligned}$$

Figure 1.11 illustrates the diagram of saddle-node bifurcations.

2. Pitchfork bifurcation

When $k_{11} = 1$ and $k_{03} = -1$ and $k_{ij} = 0$ for $(i, j) \neq ((1, 1), (i, j) \neq (0, 3))$, then system

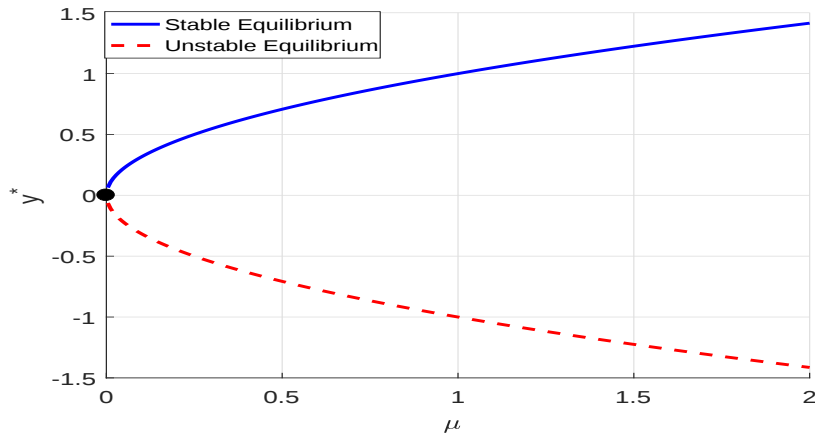


Figure 1.11: Saddle-Node Bifurcation Diagram

(1.15) become:

$$\dot{y} = (\mu y - y^3) = y(\mu - y^2) = f(y, \mu), \quad \mu \in \mathbb{R}.$$

There are three possibilities that could occur:

(a) $\mu < 0$:

$$f(y, \mu) = 0 \rightarrow \begin{cases} \mu - y^2 = 0 & \rightarrow \mu = y^2 \text{ (refuse)}, \\ \vee \\ y = 0, \end{cases}$$

then, $y = 0$ is the unique equilibrium, and it is an stable point because

$$\frac{\partial f}{\partial y}(0) = \mu < 0.$$

(b) $\mu = 0 \rightarrow f(y, \mu) = -y^3 < 0$, then, $(0,0)$ is the only equilibrium, and it is stable.

(c) $\mu > 0$:

$$f(y, \mu) = 0 \rightarrow \begin{cases} \mu - y^2 = 0 & \rightarrow y = \pm \sqrt{\mu}, \\ \vee \\ y = 0 \end{cases},$$

then, there are three equilibrium points:

$$y_1^* = -\sqrt{\mu}, y_2^* = 0, y_3^* = y = \sqrt{\mu}.$$

then:

$$\frac{\partial f}{\partial y}(y_1^*) = -2\mu < 0, \text{ then, } y_1^* \text{ is stable point.}$$

$$\frac{\partial f}{\partial y}(y_2^*) = \mu > 0, \text{ then, } y_2^* \text{ is unstable point.}$$

$$\frac{\partial f}{\partial y}(y_3^*) = -2\mu < 0, \text{ then, } y_1^* \text{ is stable point.}$$

Figure 1.12 illustrates the diagram of pitchfork bifurcations.

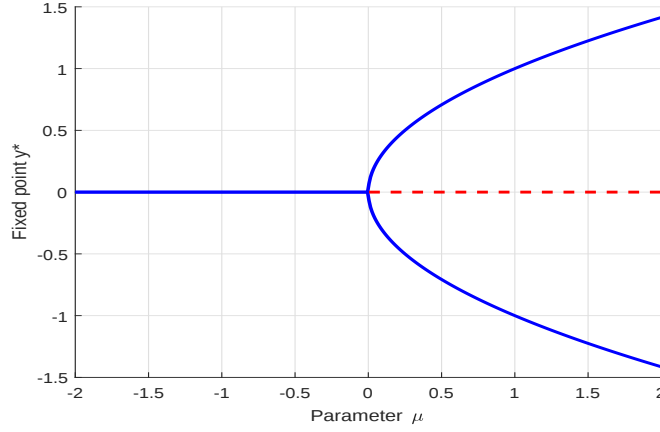


Figure 1.12: Pitchfork Bifurcation Diagram

3. Transcritical bifurcation

This kind of bifurcation in one dimension corresponds to $k_{11} = 1$ and $k_{02} = 1$ and $k_{ij} = 0$ for $(i, j) \neq ((1, 1), (i, j) \neq (0, 2))$, then system (1.15) become:

$$\dot{y} = (\mu y + y^2) = y(\mu + y) = f(y, \mu), \mu \in \mathbb{R}.$$

Two equilibrium points exist for us :

$$y_1^* = 0, y_2^* = -\mu,$$

As for the nature of equilibrium points, we have:

$$\begin{cases} \frac{\partial f}{\partial y}(y_1^*) = \mu, \\ \frac{\partial f}{\partial y}(y_2^*) = -\mu < 0. \end{cases}$$

According to the value of μ , there are three possibilities that could occur:

- (a) $\mu < 0$: $y_1^* = 0$ is stable and $y_2^* = \mu$ is unstable.
- (b) $\mu = 0$: The two points are combined into one semi-stable point.
- (c) $\mu > 0$: $y_1^* = 0$ is unstable and $y_2^* = \mu$ is stable.

Then, the stability of these fixed positions is exchanged.

The Figure 1.13 illustrates the diagram of transcritical bifurcation.

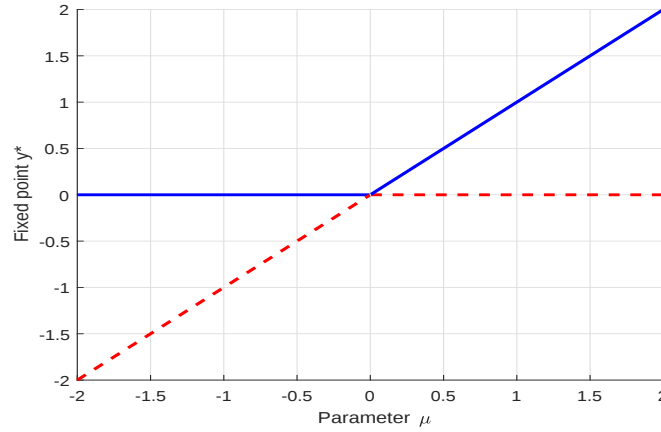


Figure 1.13: transcritical Bifurcation Diagram

1.5.2 Bifurcation in higher dimension (Hopf bifurcation)

The term "Hopf bifurcation" comes from the research of German scientist Eberhard Hopf, who investigated this type of bifurcating event in systems that are either two-dimensional or higher-dimensional. The Hopf bifurcation happens when the jacobian matrix $J_y(y^*, \mu^*)$ has two eigenvalues, which are conjugate purely imaginary, the other eigenvalues' real values different from zero, then this leads to a limit cycle and the equilibrium point loses its stability, and vice versa. Two kinds of Hopf bifurcations exists: subcritical and supercritical.

When an unstable equilibrium point produces a stable limit cycle, this is known as a supercritical Hopf bifurcation.

However, when an unstable limit cycle is generated for a stable equilibrium point, the Hopf bifurcation is called a subcritical.

Let's consider :

$$\dot{y} = f(y, \mu), \quad (1.16)$$

where $(y, \mu) \in U \subset \mathbb{R}^n \times \mathbb{R}, n > 1,$

and

$$f(y^*, \mu^*) = 0,$$

Consider the system:

$$\begin{cases} \dot{x}_1 = \mu x_1 - x_2 + p(x_1, x_2, \mu), \\ \dot{x}_2 = x_1 + \mu x_2 + q(x_1, x_2, \mu), \end{cases} \quad (1.17)$$

where (1.16) is an analytical polar system, $\mu \in \mathbb{R}$ and:

$$p(x_1, x_2, \mu) = \sum_{i+j \geq 2} a_{ij} x_1^i x_2^j.$$

$$q(x_1, x_2, \mu) = \sum_{i+j \geq 2} a_{ij} x_1^i x_2^j.$$

The following number σ :

$$\sigma = \frac{3\pi}{2} [3(a_{30} + b_{03}) + (a_{12} + b_{21}) - 2(a_{20}b_{20} + a_{02}b_{02}) + a_{11}(a_{02} + a_{20}) - b_{11}(b_{02} + b_{20})],$$

is called the Lyapunov number. According to the sign of this number, we can determine the nature of the origin as:

1. If $\sigma \neq 0$, then $(0,0)$ is a simple weak focus.
2. If $\sigma \leq 0$, then $(0,0)$ is stable.
3. If $\sigma \geq 0$, then $(0,0)$ is unstable.

Theorem 1.5.1. Consider the planar analytic system given by (1.17).

1. If $\sigma \neq 0$, a Hopf bifurcation happens at the origin, when $\mu = 0$.
2. If $\sigma \leq 0$, an unique stable limit cycle emerges from the origin of (1.17) as μ increases past zero, the Hopf bifurcation is "supercritical".
3. When $\sigma > 0$, a single unstable limit cycle emerges from the origin of planar system (1.17) as μ decreases below zero, and the Hopf bifurcation is "subcritical".

Example 1.5.1. (Example of supercritical Hopf bifurcation)

Let's the following system:

$$\begin{cases} \dot{x}_1 = \mu x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 = x_1 + \mu x_2 - x_2(x_1^2 + x_2^2), \end{cases} \quad (1.18)$$

Since the Lyapunov number of this system is

$$\sigma = -\frac{21\pi}{4} < 0,$$

then, $(0,0)$ is stable, and consequently, the system has a supercritical Hopf bifurcation in $\mu < 0$.

Using the polar coordinates: $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$

$$\begin{cases} \dot{r} = x_1\dot{x}_1 + \dot{x}_2x_2, \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}, \end{cases} \quad (1.19)$$

Replacing $\dot{x}_1$ and $\dot{x}_2$ by their values in (1.16), we get:

$$\begin{cases} \dot{r} = r(\mu - r^2), \\ \dot{\theta} = 1, \end{cases} \quad (1.20)$$

The fixed point origin $r = 0$, a stable spiral, attracts all trajectories, when $\mu < 0$.

For $\mu = 0$, the origin remains a faint but stable spiral.

For $\mu > 0$, when the origin is an unstable spiral, there exists a stable limit cycle at $r = \sqrt{\mu}$.

Figures 1.14 and 1.15 illustrate this phenomenon.

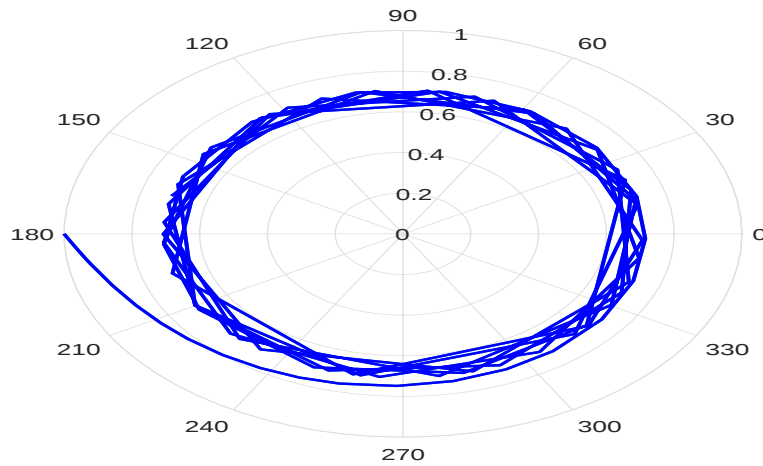


Figure 1.14: $\mu > 0$

1.6 Chaos

Over the past decade, researchers across disciplines, such as physics, biology, astronomy, and economics have developed a revolutionary approach to understanding the emergence of complexity in nature. This groundbreaking field, known as chaos theory, offers profound insights into the unpredictable and dynamic behavior of complex systems. As with many scientific concepts, chaos has no universally accepted definition, leading to the emergence of several slightly differing interpretations.

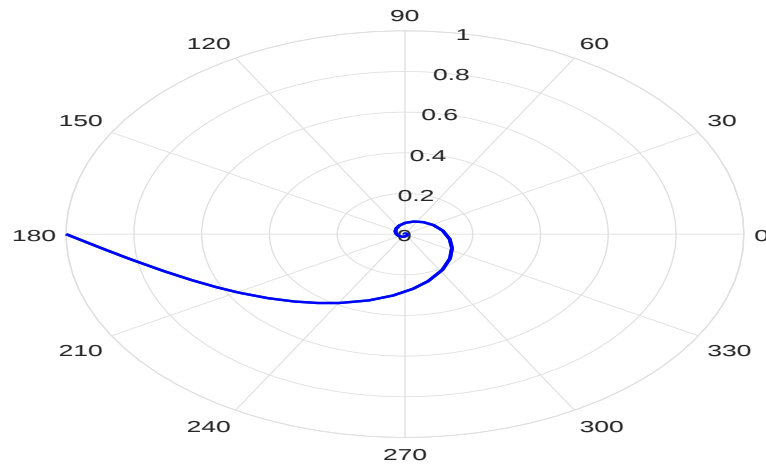


Figure 1.15: $\mu < 0$

In this section, we'll start by going over a few approaches about chaos definition, provided by different researchers, like Devaney's [26], Li-Yorke's [27], Wiggins [28], Lyapunov [28], and Knudsen [29]. Next, we'll talk about how to characterize the behavior of chaotic systems. The most significant scenarios for the path towards chaos will be discussed in the last part.

1.6.1 Devaney's approach of chaos

According to Robert L. Devaney, If a dynamical system exhibits three particular characteristics, it is considered chaotic:

1. Topological transitivity: That is, for any two arbitrary regions in the space of possible states, there exists an orbit that moves from the first region to the second region, meaning the system has a mixed nature
2. Dense periodic orbits.
3. Sensitivity to starting conditions (property of the "butterfly effect"): A small perturbation in the initial states leads to different outcomes, making long-term prediction practically impossible.

1.6.2 Li-Yorke's approach of chaos

According to Li and Yorke, the function $g \in C^n$ is said to be Li-Yorke chaotic, if there exists a pair of different initial conditions x_1 and x_2 , such that, for some $\epsilon > 0$:

$$\lim_{n \rightarrow \infty} \sup |g^n(x_1) - g^n(x_2)| > 0,$$

which means that the system shows the property of sensitivity to initial conditions.

In addition, the system must have dense periodic orbits and comes arbitrarily close to every state in the state space (property of topologically mixing).

1.6.3 Wiggins' approach chaos

Wiggins' approach of chaos focuses on the following characteristics:

1. Topological transitivity
2. Dense periodic orbits.
3. Sensitivity to initial conditions.
4. Strange attractor which has a non-integer dimension.

1.6.4 Lyapunov's approach of chaos

Lyapunov's approach of chaos is focussed on the concept of Lyapunov exponents, which measure the rate of separation of two close trajectories in any dynamical system. A dynamical system is classified as chaotic if it fulfills the following fundamental condition:

- * At least a positive Lyapunov exponent is required for the system.

1.6.5 Knudsen's approach of chaos

Let the continuous function $H : E \rightarrow E$ be defined on the metric space (E, d) ; then according to Knudsen, the dynamical system is chaotic if:

1. H owns a dense orbit.
2. The sensitivity of H is dependent on the initial conditions.
3. It is driven by deterministic laws, meaning no random processes are involved, while its sensitivity to initial conditions makes its behavior appear unpredictable over time.

1.6.6 Some characteristics of chaotic systems

Chaotic systems are nonlinear deterministic systems that are complex and highly sensitive to initial conditions. Here are the most important characteristics of chaotic systems :

A) Sensitivity to initial conditions

A small disturbance in the initial state of a nonlinear system can lead to very different behaviors (a property of the butterfly effect).

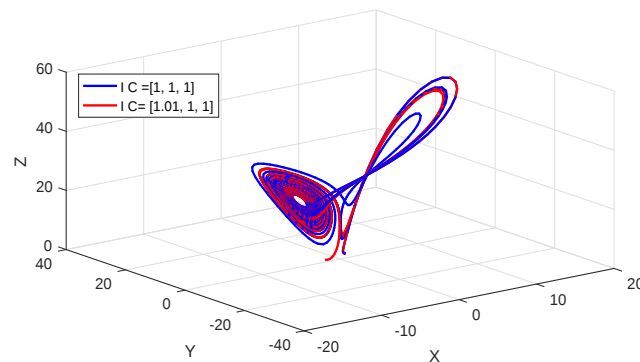


Figure 1.16: Sensitivity to initial conditions of Lorenz system

B) Determinisme

It is driven by deterministic laws, meaning no random processes are involved, while its sensitivity to initial conditions makes its behavior appear unpredictable over time.

C) Lyapunov exponents

Sensitivity is quantified using the Lyapunov exponent. A positive Lyapunov means that the system is highly responsive to the starting conditions. According to this, Lyapunov exponents quantify how quickly orbits diverge from one another. The number of Lyapunov exponents is equivalent to the dimension of the phase space, and they are typically arranged from greatest to smallest $\lambda_1, \lambda_2, \lambda_3, \dots$. Three requirements must be met by the Lyapunov exponents in order for chaos to appear:

- * At least one of them is positive, which explains why the trajectories diverge.
- * The trajectories converge, since at least one of them is negative.

* The dissipative nature of a chaotic system, or its energy loss, is explained by the fact that the sum of all the exponents is negative.

D) Strange attractor

Chaotic systems do not follow favored trajectories and are, at all scales, very unpredictable and infinitely complicated. The curve of such system always evolves in a bounded space, ending up describing a certain geometric object, that symbolizes its attractor. This behavior is dubbed weird, since it never passes through the same places.

E) Fractal structure

In general, chaotic or hyperchaotic systems exhibit self-similarity at different scales, creating fractal-like patterns when visualized.

F) Long-term unpredictability

Although a chaotic system is deterministic, it has the property of unpredictability due to its sensitivity to initial conditions.

G) Non-periodic behavior

A chaotic or hyperchaotic system has an unpredictable behavior that never repeats. Their behavior is aperiodic and irregular.

1.6.7 Scenario towards chaos

The classic scenario towards chaos or hyperchaos refers to the process by which a deterministic dynamic system transitions from stable behavior to chaotic behavior. This process is typically observed in nonlinear systems, where small disturbances in

initial conditions can lead to entirely different outcomes, which is one of the most important characteristics of chaos.

1. Period doubling

Period doubling is a fundamental route to chaos observed in nonlinear dynamical systems, where a system undergoes a sequence of bifurcations, each time doubling the period of its limit cycle.

As a system parameter is varied, the system transitions from:

- A stable fixed point, to
- A period-2 cycle, then
- A period-4 cycle, then period-8, and so on,

Eventually leading to chaotic behavior.

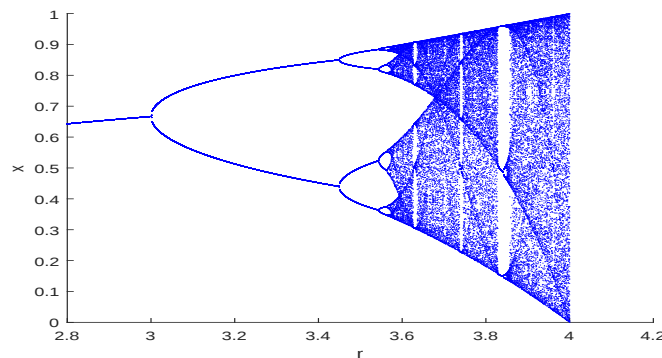


Figure 1.17: Period doubling of Logistic map.

2. Intermittence

The unpredictable development of chaotic bursts in a regularly oscillating system is what defines this situation via intermittences. After maintaining a certain "regularity"—a periodic or almost periodic regime—for a while, the system suddenly destabilizes, resulting in a chaotic explosion.

3. Quasi-periodicity

Quasi-periodicity, which happens when a second system upsets an initially periodic system, is the third scenario of the transition to chaos. A system is considered quasi-periodic if the ratio of the periods of the two systems is not rational. As

a result, this regime may become unstable and either directly chaotic or a third frequency may appear.

Fractional calculus

Fractional calculus is one of the most crucial tools in the science of non-linear dynamic systems. Integer-order integration and differentiation operations are extended by it. .

An equation with fractional derivatives is called a fractional differential equation. These equations can be conveniently and correctly modeled for many systems, such as heat diffusion systems, viscoelastic systems, batteries, etc. By including fractional-order derivatives, fractional differential equations (FDEs) expand on the idea of ordinary differential equations and partial differential equations. These equations provide a solid framework for simulating processes that are difficult to capture with conventional integer-order differential equations, such as memory, hereditary characteristics, or anomalous dynamics. Due to their ability to account for memory and non-locality, they have become an essential tool in contemporary research and engineering, driving advancements in theoretical and applied fields.

In this chapter, we discuss some of the most important elements associated with fractional calculus. In the first section, we recall some important spatial functions such as Gamma, Beta and Mittag-Leffler functions. The second section includes Laplace's transform with some of its properties. The third section contains a set of the most famous definitions concerning fractional differential operators, namely those of Riemann-Liouville, Caputo and Grünwald-Letnikov .

2.1 Special functions

In this part, we give some concepts about three types of special functions: Gamma, Beta, and Mittag-Leffler, which are essential to the study of fractional-order differential equations.

2.1.1 Gamma function

The gamma function of Euler [30, 31], which is denoted by $\Gamma(\cdot)$, is among the most crucial roles in fractional calculus. The definition of this function given by the Euler limit is as follows:

$$\Gamma(x) = \lim_{S \rightarrow \infty} \left[\frac{S! S^x}{x[x+1][x+2] \dots [x+S]} \right]. \quad (2.1)$$

An other definition of this function is:

$$\Gamma(x) = \int_0^{\infty} y^{x-1} e^{-y} dy, x > 0. \quad (2.2)$$

The most crucial characteristic of the gamma function is:

$$\Gamma(x+1) = x\Gamma(x). \quad (2.3)$$

If $\Gamma(1) = 1$, we get:

$$\Gamma(n+1) = n\Gamma(n) = n.(n-1)! = n!.$$

According to (2.2), the gamma function is just for strictly positive numbers. In the following, we will see how to extend it to negative numbers:

We have:

$$\Gamma(x-1) = \frac{\Gamma(x)}{x-1}, \text{ for } -1 < x-1 < 0,$$

$$\Gamma(x-2) = \frac{\Gamma(x-1)}{x-2}, \text{ for } -2 < x-2 < -1,$$

.
. .
.

Finally:

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \text{ for } -n < x < -(n-1).$$

Figure 2.1 represents the graph of gamma function .

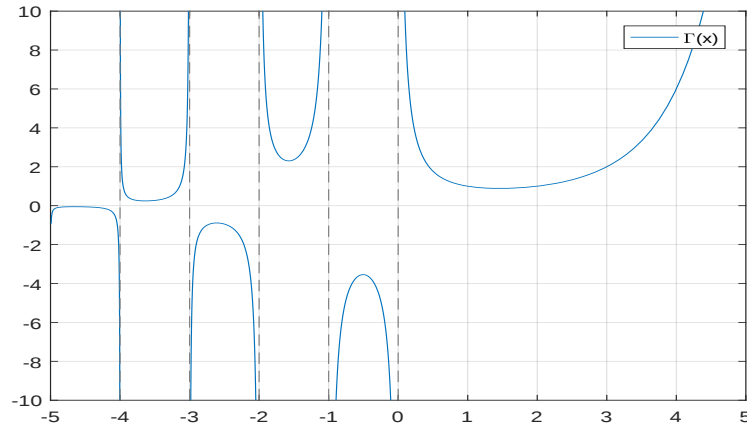


Figure 2.1: Graph of gamma function

2.1.2 Beta function

The beta function is defined as follows [30]:

$$B(z, w) = \int_0^1 \tau^{z-1} (1 - \tau)^{w-1} d\tau, \text{ for } \operatorname{Re}(z) > 0 \text{ and } \operatorname{Re}(w) > 0. \quad (2.4)$$

The relationship between the gamma and beta functions is :

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}.$$

Figure 2.2 represents the beta's graph:

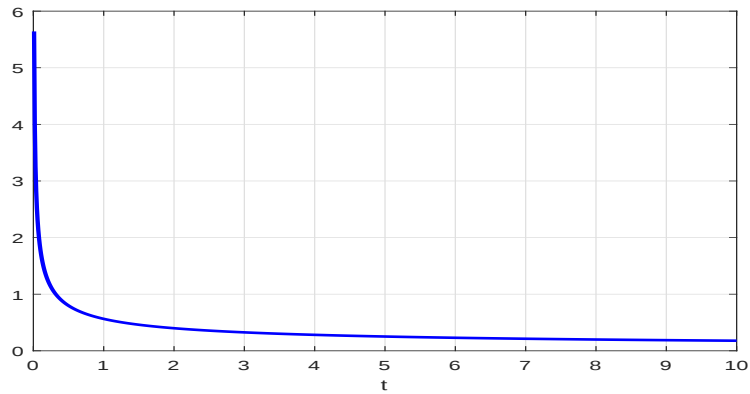


Figure 2.2: Graph of beta function

2.1.3 Mittag-Leffler function

To solve any differential equation in the theory of differential equations in integer order, the exponential function $\exp(\cdot)$ is necessary. In fractional-order differential equations,

the Mittag-Leffler (sometimes also called the generalized exponential function) plays the same role. It had been introduced by G.M. Mittag-Leffler [30], and it is described by:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (2.5)$$

The function of Mittag-Leffler of two-parameter type is provided by:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0. \quad (2.6)$$

As an example:

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 2)} = \frac{e^z - 1}{z}, \quad (2.7)$$

Figure 2.3 represents the Mittag-leffler's graph.

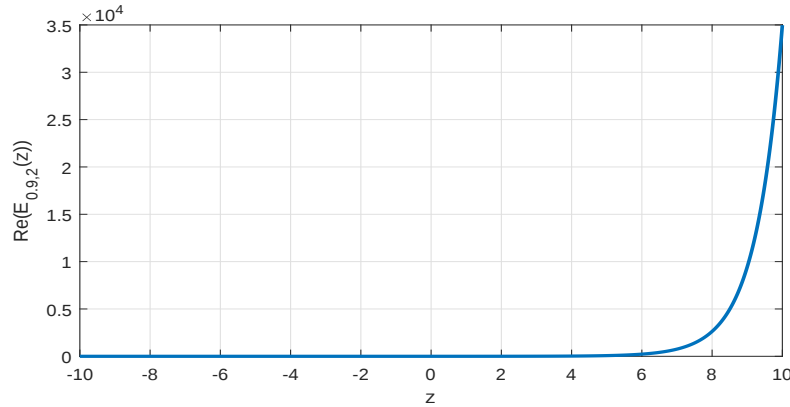


Figure 2.3: Mittag-leffler's Graph

Derivation of Mittag-Leffler function

* The function of Mittag-Leffler with one-parameter

The Mittag-Leffler function's m -th derivation using the parameter α is :

$$E_{(\alpha)}^{(m)} = \sum_{k=0}^{\infty} \frac{(k+m)!z^k}{k!\Gamma(\alpha k + \alpha m + 1)}.$$

* The function of Mittag-Leffler with two-parameters

The m -th derivation of the function of Mittag-Leffler with two parameters α and β is:

$$E_{(\alpha,\beta)}^{(m)} = \sum_{k=0}^{\infty} \frac{(k+m)!z^k}{k!\Gamma(\alpha k + \alpha m + \beta)}.$$

2.2 Laplace transform

In fractional calculus theory, the Laplace transform approach is crucial, particularly when resolving fractional differential equations.

Here, we define the Laplace transform and discuss some of its features.

It's important to remember that any function $f(\cdot)$ is considered to be of exponential order β , if there are two constants $N > 0$ and $T > 0$, such that:

$$e^{\beta t}|f(t)| \leq N, \text{ for all } t \geq T. \quad (2.8)$$

Consider a function $f(t)$ having exponential order β , then [32]:

$$\int_0^{\infty} f(t)e^{-st} dt, \quad (2.9)$$

exist for all s , where $\text{Re}(s) > \beta$.

The integral (2.9) is known as the function $f(t)$ Laplace transform, and we identify it by $L\{f(t)\}$, then:

$$L\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt. \quad (2.10)$$

$f(t)$ is called the origin and it can be restored from the integral:

$$f(t) = L\{F(t); t\} = \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \text{ for } c = \text{Re}(s) > c_0, \quad (2.11)$$

with the Laplace integral's absolute convergence is located in the right half plane by c_0 .

Example 2.2.1. *Observe the functions:*

$$f(t) = \exp(at), \text{ and } g(t) = t^u,$$

where $u > -1$ and $a, t \in \mathbb{R}$.

The Laplace transform of the considered functions are

$$L\{t^u\} = \frac{\Gamma(u+1)}{s^{u+1}}, \text{ and } L\{e^{at}\} = \frac{1}{s-a}.$$

Theorem 2.2.1.

Let $\beta > 0$, $\varphi \in \mathbb{C}$, define $y(t) = E[-\beta(\varphi t^\beta)]$. The Laplace transform of $y(t)$ is:

$$Ly(s) = \frac{s^{\beta-1}}{s^\beta + \varphi}, \quad (2.12)$$

where $\varphi \in \mathbb{C}$, $\text{Re}(s) > 0$, and $|\varphi s^{-\beta}| < 1$.

Remark 2.2.1.

Differentiating equation (5.13) n times with respect to φ produces the next relationship:

$$L\left\{t^{\beta n} E^{(n)}(-\varphi t^{\beta})\right\}(s) = \frac{n!(s^{\beta-1})}{(s^{\beta} + \varphi)^{n+1}} \quad (2.13)$$

where $\varphi \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, and $|\varphi s^{-\beta}| < 1$.

Theorem 2.2.2. (Theorem of final value)

Assume that $F(s)$ is the Laplace transformation of the function $f(t)$. Consider $F(s)$ as Laplace transformation of the function $f(t)$. If the open left half-plane contains all of the poles of $sF(s)$, we have

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s). \quad (2.14)$$

Laplace transform properties

Important characteristics of the Laplace transform include:

*** The product convolution's Laplace transform:**

Let $H(s)$ be the Laplace transforms of $h(t)$ and $K(s)$ be the Laplace transforms of $k(t)$, then the Laplace transform of $h(t)$ and $k(t)$'s product convolution is provided by:

$$L\{h(t) * k(t); s\} = H(s)K(s). \quad (2.15)$$

Example 2.2.2. Let: $f(t) = e^{-t}$ and $g(t) = \sin(t)$,

Our goal is to compute the Laplace Transform of $(f * g)(t)$, the Laplace transforms of $f(t)$ and $g(t)$ are :

$$L\{e^{-t}\} = \frac{1}{s+1}, L\{g(t)\} = \frac{1}{1+s^2}.$$

Based on the theorem of convolution, we have:

$$L\{e^{-t} * \sin(t)\} = \left(\frac{1}{s+1}\right) \cdot \left(\frac{1}{1+s^2}\right) = \frac{1}{(1+s)(1+s^2)}.$$

*** The derivative's Laplace transform:**

Let n be an integer number. For the Laplace transform of the function $g(t)$ derivative is:

$$L\{h^{(n)}(t); s\} = s^n H(s) - \sum_{k=0}^{n-1} s^{n-k-1} h^{(k)}(0). \quad (2.16)$$

* **Fractional integral's Laplace transform:**

Consider the next fractional integral of $y(t)$:

$$D^{-\alpha}(y) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} y(z) dz, \alpha > 0. \quad (2.17)$$

This expression is a convolution integral.

Then :

$$L\{D^{-\alpha}y(t)\} = \frac{1}{\Gamma(\alpha)} L\{t^{\alpha-1}\} L\{y(t)\} = s^{-\alpha} Y(t), \quad (2.18)$$

with $Y(t)$ is the Laplace transform of $y(t)$.

2.3 Fractional derivatives

Fractional derivatives is an ancient subject, returning to a group of mathematician researchers like Leibniz, Caputo, and Riemann–Liouville. This topic has gained great importance due to its utilization in many sciences and engineering fields, like data encryption, biomedical engineering, and secure communication.

The derivation with fractional-order is a generalization of the classical derivative. Since 1695, many researchers have attempted to give some approach definitions about it. Here, we will mention the most popular of them like Caputo, Riemann-Liouville and Grünwald-Letnikov.

2.3.1 Grünwald-Letnikov approach of fractional-order derivative

This technique is defined by Grünwald and Letnikov [33]. They defined it by finite fractional differences.

Let us consider the relation:

$$f^{(n)}(x) = \lim_{h \rightarrow 0} \frac{(\Delta_h^n f)(x)}{h^n}, \quad (2.19)$$

$(\Delta_h^n f)(x)$ is a finite difference of order $n \in \mathbb{N}$ and a step $h \in \mathbb{R}$, which is defined by:

$$(\Delta_h^n f)(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x - kh), \text{ for } x \in \mathbb{R}. \quad (2.20)$$

Replacing $n \in \mathbb{N}$ in (2.19) by $\alpha > 0$, we get:

$${}_a D_t^\alpha f(x) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} f(x - kh), \quad (2.21)$$

where $nh = t - \alpha$.

The expression (2.21) is called Grünwald-Letnikov approach of fractional derivative of f .

2.3.2 Riemann-Liouville approach of fractional derivative

Let f be an integrable and continuous function in every finite interval (a, t) .

The formula of Cauchy for the $m - th$ integral of f is:

$${}_a D_t^{-m} f(t) = \frac{1}{(m-1)!} \int_a^t \frac{f(\zeta)}{(t-\zeta)^{1-m}} d\zeta, \quad (2.22)$$

Using Gamma function, we have:

$${}_a D_t^{-m} f(x) = \frac{1}{\Gamma(m)} \int_a^t \frac{f(\zeta)}{(t-\zeta)^{1-m}} d\zeta, \quad (2.23)$$

The concept behind of the integral of Riemann-Liouville with fractional-order is to generalize Cauchy's formula for obtaining $n - th$ integration to a non-integer order. To this end, we replace n by a real $\alpha > 0$. Then the fractional integral of Riemann-Liouville [30] is provided by:

$${}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\zeta)}{(t-\zeta)^{1-\alpha}} d\zeta, \quad (2.24)$$

The Riemann-Liouville derivative fractional of function $f(t)$ is:

$${}_a D_t^\alpha f(t) = \frac{d^m}{(dt)^n} ({}_a D_t^{-(n-\alpha)} f(t)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{(dt)^m} \int_a^t \frac{f(\zeta) d\zeta}{(t-\zeta)^{\alpha+1-m}}, \quad (2.25)$$

where $m-1 < \alpha \leq n$.

2.3.3 Caputo Approach of fractional derivative

Now, suppose that f is of class C^n , $n \in \mathbb{N}$. So, for $n \geq 1$, the fractional derivative of Caputo of the function $f(t)$ [34] is given by:

$${}_a^C D_t^\alpha f(t) = {}_a^C D_t^{-(n-\alpha)} \left(\frac{d^n}{(dt)^n} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\zeta) d\zeta}{(t-\zeta)^{\alpha+1-n}}, \quad (2.26)$$

where $n - 1 < \alpha \leq n$.

Caputo's fractional derivative: some characteristics

- * There is a relationship between the fractional derivatives of Riemann-Liouville and fractional derivatives of Caputo . The two approaches are linked together by the next relationship:

$${}_a^c D_t^\alpha(t) = {}_a D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k - \alpha + 1)} (t - a)^{k-\alpha}, n - 1 < \alpha \leq n. \quad (2.27)$$

- * The Caputo's differential operator has a property of linearity. i.e., for $(\eta, \mu) \in \mathbb{R}$:

$${}_a^c D_t^\alpha(\eta x(t) + \mu y(t)) = \eta {}_a^c D_t^\alpha x(t) + \mu {}_a^c D_t^\alpha y(t).$$

- * Let $f(t) = c$. Since $f^{(n)} = 0$, we get:

$${}_a^c D_t^\alpha c = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(\zeta)}{(t - \zeta)^{\alpha-n+1}} d\zeta = 0. \quad (2.28)$$

- * The Caputo's differential operator has the following property

$${}_a^c D_t^\alpha ({}_a^c D_t^m f(t)) = {}_a^c D_t^{\alpha+m} f(t), \quad (2.29)$$

where $m \in \mathbb{N}^*$ and $n - 1 < \alpha \leq n$.

- * The fractional derivation of the function $f(t) = (t - a)^\beta$ is:

$${}_a^c D_t^\alpha (t - a)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} (t - a)^{\beta-\alpha}, \text{ for all } \beta > \alpha. \quad (2.30)$$

if $\beta = 2$ and $a = 0$, the fractional derivative of $f(t) = t^2$ is:

$${}_0^c D_t^\alpha t^2 = \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha}. \quad (2.31)$$

Figure 2.4 illustrate the graph of t^2 , for different values of α .

- * The Caputo fractional derivative's Laplace transformation is:

$$L \{D^\alpha f(t), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \quad (2.32)$$

with $n - 1 \leq \alpha < n$.

if $\alpha \in (0, 1]$, we have

$$L \{D^\alpha f(t), s\} = s^\alpha F(s) - s^{\alpha-1} f(0) \quad (2.33)$$

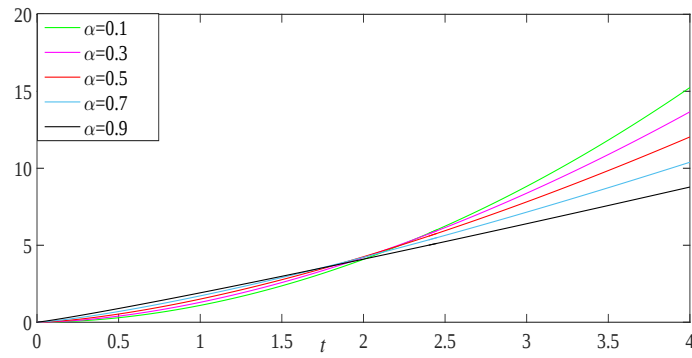


Figure 2.4: Graph of the fonction t^α , for differents values of α .

The Laplace transform is often used to find the solution of differential equations; it's necessary to choose which type of initial conditions to use. Recall that the Laplace transform of the Caputo fractional derivative enables the use of same initial values of integer-order, leading to a physical interpretation. For this, it will be frequently employed, throughout this work.

Fractional dynamical systems

A fractional dynamical system is a system that includes fractional-order derivatives instead of integer-order derivatives. The researchers use these systems for simulating real-world processes of memory effects, hereditary characteristics, and anomalous diffusion.

In this chapter, Caputo differential equations with fractional-order and the existence and uniqueness of a solution theorem, as well as the numerical solutions and analytical solutions of linear equations and non-linear equations are covered. Moreover, we will talk about the qualitative study of the dynamical system with Caputo fractional derivative, including fixed points and the stability in both cases commensurate systems with fractional-order and non-commensurate systems with fractional-order.

3.1 Fractional differential equation

Our interest in this part of the chapter is limited to the important elements of the differential equations with fractional-order theory of Caputo fractional derivative. We recall the theorem of existence and uniqueness, the impact of solving a fractional differential equation's with initial value problem with Caputo fractional derivative, then the explicit resolution of linear fractional differential equations, and we end this part by presenting a numerical technique necessary for the resolution of fractional equations.

Consider the initial value problem:

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)) \\ {}^C D^j y(0) = y_0^{(j)}, \quad j = 0, 1, \dots, n-1, \end{cases} \quad (3.1)$$

with ${}^C D_t^\alpha$ indicates the derivation operator in the Caputo sense, and $n-1 < \alpha \leq n$.

The existence and uniqueness of the solution to the initial value problem (3.1) can be established by the The existence and uniqueness of the solution to the initial value problem (3.1) can be established by the following theorem.

Theorem 3.1.1. [35]

Let $R > 0$, $k^* > 0$ and $y_0^{(j)} \in \mathbb{R}$, $j = 0, 1, \dots, m-1$, and let $g : S = [0, h^*] \times \mathbb{R} \longrightarrow \mathbb{R}$, be a continuous function, satisfying the Lipschitz condition for y :

$$|g(t, y_1) - g(t, y_2)| < L|y_1 - y_2|, \quad (3.2)$$

and let:

$$k = \min\{k^*, (R\Gamma(\alpha + 1)/N)^{1/\alpha}\}, \quad (3.3)$$

where:

$$N = \sup_{t, z \in k} |g(t, z)|, \quad (3.4)$$

then, $y \in C[0, h]$ is the unique solution of the problem (3.1).

Theorem 3.1.2. [35]

Under the assumptions of the 3.1.1 theorem, the initial conditions problem (3.1) is equivalent to the equation of Volterra integral :

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (3.5)$$

Proof. Suppose that y represents a solution of equation (3.5). The latter can be given as:

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + {}_0 D_t^{-\alpha} f(t, y(t)). \quad (3.6)$$

Applying the differentiation operator ${}_0^C D_t^\alpha$ on both parts of this relation, we find:

$$\begin{aligned} {}_0^C D_t^\alpha y(t) &= \sum_{k=0}^{n-1} \frac{y_0^{(k)} {}_0^C D_t^\alpha t^k}{k!} + {}_0^C D_t^\alpha {}_0 D_t^{-\alpha} f(t, y(t)) \\ &= \sum_{k=0}^{n-1} \frac{y_0^{(k)} {}_0^C D_t^\alpha t^k}{k!} + f(t, y(t)). \end{aligned} \quad (3.7)$$

Since $k < \alpha$, then ${}_0^C D_t^\alpha t^k = 0$

Therefore y is a solution of the equation (3.1).

It remains to show that: ${}_0^C D_t^j y(0) = y_0^{(j)}$.

Indeed, by applying the operator ${}_0^C D_t^j$, $0 \leq j \leq n-1$ on the Volterra equation (3.5), we obtain:

$$\begin{aligned} {}_0^C D_t^j y(t) &= \sum_{k=0}^{n-1} \frac{y_0^{(k)} {}_0^C D_t^j t^k}{k!} + {}_0^C D_t^j {}_0 D_t^{-j} {}_0 D_t^{-(\alpha-j)} f(t, y(t)) \\ &= \sum_{k=0}^{n-1} \frac{y_0^{(k)} {}_0^C D_t^j t^k}{k!} + {}_0 D_t^{-(\alpha-j)} f(t, y(t)). \end{aligned} \quad (3.8)$$

We remind that:

$${}_0^C D_t^j t^k = \begin{cases} 0, & \text{si } j > k \\ \Gamma(k+1), & \text{if } j = k \\ \frac{\Gamma(k+1)}{\Gamma(k-j+1)} t^{k-j}, & \text{if } j < k. \end{cases} \quad (3.9)$$

Which implies:

$${}_0^C D_t^j t^k \Big|_{t=0} = \begin{cases} 0, & \text{si } j > k \\ \Gamma(k+1), & \text{if } j = k \\ 0, & \text{if } j < k. \end{cases} \quad (3.10)$$

However $\alpha - j \geq 1$.

Which indicates that the integral ${}_0 D_t^{-(\alpha-j)} f(t, y(t))|_{t=0}$ is null.

Therefore ${}_0^C D_t^j y(0) = y_0^{(j)}$.

Let us now suppose that y is a solution of equation (3.1) and show that y is the solution to (3.5).

Let us set $z(t) = f(t, y(t))$, we then have:

$$\begin{aligned} z(t) &= f(t, y(t)) = {}_0^C D_t^\alpha y(t) = {}_0^R D_t^\alpha y(t) - \sum_{k=0}^{n-1} \frac{y_0^{(k)} t^{k-\alpha}}{\Gamma(k-\alpha+1)} \\ &= {}_0^R D_t^\alpha y(t) - {}_0^R D_t^\alpha \sum_{k=0}^{n-1} \frac{y_0^{(k)} t^k}{k!} \\ &= {}_0^R D_t^\alpha (y(t) - \sum_{k=0}^{n-1} \frac{y_0^{(k)} t^k}{k!}) \\ &= {}_0^R D_t^\alpha (y - T_{n-1}[y, 0])(t) \\ &= {}_0 D_t^n {}_0 D_t^{-(n-\alpha)} (y - T_{n-1}[y, 0])(t), \end{aligned} \quad (3.11)$$

with $T_{n-1}[y, 0](t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)}$, is the Taylor polynomial with degree $n - 1$.

By applying the operator ${}_0D_t^{-n}$ on both sides of this last equation, we get:

$${}_0D_t^{-n}z(t) = {}_0D_t^{-(n-\alpha)}(y - T_{n-1}[y, 0])(t) + q(t), \quad (3.12)$$

q represents a polynomial with degree $\leq n - 1$.

Because the function z is continuous, the function ${}_0D_t^{-n}z$ has a zero with order at least n at the origin.

Furthermore, using construction : $y - T_{n-1}[y, 0]$ having the similar property .

therefore the function ${}_0D_t^{-(n-\alpha)}(y - T_{n-1}[y, 0])$ must also have a zero with order n .

Consequently the polynomial q having the similar property, but since it is of degree $\leq n - 1$, it follows that $q = 0$.

Consequently:

$${}_0D_t^{-n}z(t) = {}_0D_t^{-(n-\alpha)}(y - T_{n-1}[y, 0])(t). \quad (3.13)$$

that implies:

$$\begin{aligned} y(t) &= T_{n-1}[y, 0](t) + {}_0D_t^{-\alpha}z(t) \\ &= \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \end{aligned} \quad (3.14)$$

□

3.1.1 Analytical resolution of fractional linear equations

In this part, we are interested in explicitly solving a linear fractional differential equation with Caputo derivative .

A one-dimensional equation case

Theorem 3.1.3. [36]

Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. The general form of the solution to the problem:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = \lambda y(t) + q(t), & \lambda \in \mathbb{R}, \text{ and } q \text{ a continuous function.} \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \dots, n - 1, \end{cases} \quad (3.15)$$

is given by the formula:

$$y(t) = \sum_{k=0}^{n-1} y_0^{(k)} u_k(t) + \tilde{y}(t), \quad (3.16)$$

where:

$$\tilde{y}(t) = \begin{cases} {}_0D_t^{-\alpha}q(t), & \text{if } \lambda = 0 \\ \frac{1}{\lambda} \int_0^t q(t-\tau)u'_0(\tau)d\tau, & \text{if } \lambda \neq 0, \end{cases} \quad (3.17)$$

and:

$$u_k(t) = D^{-k}e_\alpha(t), \quad \text{where } e_\alpha(t) = E_\alpha(\lambda t^\alpha), \quad k = 0, 1, \dots, n-1. \quad (3.18)$$

Proof. • Si $\lambda = 0$, the problem (3.15) becomes:

$$\begin{cases} {}^C_0D_t^\alpha y(t) = q(t) \\ y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \dots, n-1. \end{cases}$$

Because $e_\alpha(t) = E_\alpha(0) = 1$, we have $u_k(t) = \frac{t^k}{k!}$, pour tout $k = 0, 1, \dots, n-1$.

Using the relationship between the Riemann-Liouville derivative and that of Caputo, we find:

$${}^C_0D_t^\alpha y(t) = {}^R_0D_t^\alpha y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} = q(t). \quad (3.19)$$

Which implies:

$${}^R_0D_t^\alpha y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} + q(t). \quad (3.20)$$

We apply the integral of Riemann-Liouville on both parties of the equation (3.20), we find:

$$\begin{aligned} {}_0D_t^{-\alpha} {}^R_0D_t^\alpha y(t) &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0) {}_0D_t^{-\alpha} t^{k-\alpha}}{\Gamma(k-\alpha+1)} + {}_0D_t^{-\alpha} q(t) \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0) t^k}{k!} + {}_0D_t^{-\alpha} q(t). \end{aligned} \quad (3.21)$$

The latter is equivalent to:

$$y(t) = \sum_{k=0}^{n-1} y^{(k)}(0) u_k(t) + \tilde{y}(t), \quad \text{where } \tilde{y}(t) = {}_0D_t^{-\alpha} q(t). \quad (3.22)$$

• Now suppose that $\lambda \neq 0$. The proof is done in two steps (a) and (b).

a) The function u_k satisfies The differential equation that is homogeneous, that is:

${}^C_0D_t^\alpha u_k = \lambda u_k, \forall k = 1, \dots, n-1$ and verifies the initial conditions $u_k^{(j)}(0) = \delta_{kj}$ (Kronecker delta) for $j, k = 0, \dots, n-1$.

b) The function $\tilde{y}$ represents a solution of the non-homogeneous differential equation,

with The differential equation that is homogeneous.

We start with (a).

A simple calculation shows that:

$$e_\alpha(t) = E_\alpha(\lambda t^\alpha) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j}}{\Gamma(\alpha j + 1)}. \quad (3.23)$$

Hence:

$$u_k = D^{-k} e_\alpha(t) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j + k}}{\Gamma(\alpha j + 1 + k)}. \quad (3.24)$$

Let us now show that u_k is a solution of the homogeneous differential equation. Indeed:

$$\begin{aligned} {}^C_0 D_t^\alpha u_k(t) &= {}^C_0 D_t^\alpha \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j + k}}{\Gamma(\alpha j + 1 + k)} \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + 1 + k)} {}^C_0 D_t^\alpha t^{\alpha j + k} \\ &= \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(\alpha(j-1) + 1 + k)} t^{\alpha(j-1) + k} \\ &= \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + 1 + k)} t^{\alpha j + k} \\ &= \lambda u_k(t). \end{aligned} \quad (3.25)$$

Which ensures that u_k represents a solution of the homogeneous equation.

Furthermore, if $j = k$, we have:

$$u_k^{(k)}(0) = D^k D^{-k} e_\alpha(0) = e_\alpha(0) = 1. \quad (3.26)$$

If $j < k$:

$$u_k^{(j)}(0) = D^j D^{-k} e_\alpha(0) = D^{-(k-j)} e_\alpha(0) = 0. \quad (3.27)$$

And for $j > k$, we get:

$$u_k^{(j)}(0) = D^j D^{-k} e_\alpha(0) = D^{(j-k)} e_\alpha(0) = 0. \quad (3.28)$$

However, in contrast:

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{\lambda} \int_0^t q(t-\tau) u'_0(\tau) d\tau \\ &= \frac{1}{\lambda} \int_0^t q(t-\tau) e'_\alpha(\tau) d\tau \\ &= \frac{1}{\lambda} \int_0^t q(t) e'_\alpha(t-\tau) d\tau. \end{aligned} \quad (3.29)$$

We recall that this integral always exists, whatever $t > 0$, because q is a continuous function, e'_α integrable, and $\tilde{y}(0) = 0$.

According to a property of the differentiation of integrals with a parameter, we have:

$$D\tilde{y}(t) = \frac{1}{\lambda} \int_0^t q(\tau) e''_\alpha(t - \tau) d\tau + \frac{1}{\lambda} q(t) \underbrace{e'_\alpha(0)}_{=0}. \quad (3.30)$$

Using the property of continuity of q and the weak singularity of e''_α , we can easily verify that $D^{(1)}\tilde{y}(0) = 0$.

Similarly, we also find:

$$D^k \tilde{y}(t) = \frac{1}{\lambda} \int_0^t q(\tau) e_\alpha^{(k+1)}(t - \tau) d\tau, \quad \text{for all } k = 0, \dots, n-1. \quad (3.31)$$

Thus, $D^k \tilde{y}(0) = 0$.

As a result $\tilde{y}$ satisfies all initial conditions that are homogeneous.

Let us now show that $\tilde{y}$ represents a solution of the non-homogeneous differential equation.

Since:

$$e'_\alpha(u) = \frac{d}{du} e_\alpha(u) = \sum_{j=1}^{\infty} \frac{\lambda^j u^{\alpha j-1}}{\Gamma(\alpha j)}. \quad (3.32)$$

So, we get:

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{\lambda} \int_0^t q(\tau) e'_\alpha(t - \tau) d\tau \\ &= \frac{1}{\lambda} \int_0^t q(\tau) \sum_{j=1}^{\infty} \frac{\lambda^j (t - \tau)^{\alpha j-1}}{\Gamma(\alpha j)} d\tau \\ &= \sum_{j=1}^{\infty} \lambda^{j-1} \frac{1}{\Gamma(\alpha j)} \int_0^t q(\tau) (t - \tau)^{\alpha j-1} d\tau \\ &= \sum_{j=1}^{\infty} \lambda^{(j-1)} {}_0D_t^{-\alpha j} q(t). \end{aligned} \quad (3.33)$$

Thus:

$$\begin{aligned}
{}_0^C D_t^\alpha \tilde{y}(t) &= \sum_{j=1}^{\infty} \lambda^{j-1} {}_0^C D_t^\alpha {}_0 D_t^{-\alpha j} q(t) \\
&= \sum_{j=1}^{\infty} \lambda^{j-1} {}_0 D_t^{-\alpha(j-1)} q(t) \\
&= \sum_{j=0}^{\infty} \lambda^j {}_0 D_t^{-\alpha j} q(t) \\
&= q(t) + \sum_{j=1}^{\infty} \lambda^j {}_0 D_t^{-\alpha j} q(t) \\
&= q(t) + \lambda \tilde{y}(t).
\end{aligned} \tag{3.34}$$

□

Example 3.1.1.

As an example, we take:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = y(t) - 2, \alpha \in (0, 1), \\ y(0) = 0, y'(0) = 0 \end{cases} \tag{3.35}$$

Since : $\lambda = 1$ and $q(t) = -2$, one can gets:

$$\tilde{y}(t) = \frac{1}{\lambda} \int_0^t q(t - \tau) u'_0(\tau) d\tau \tag{3.36}$$

$$\begin{aligned}
&= \int_0^t -2 u'_0(\tau) d\tau \\
&= -2 [E_\alpha(\tau^\alpha)]_0^t \\
&= 2 - 2E_\alpha(t^\alpha).
\end{aligned} \tag{3.37}$$

On the other hand:

$$\sum_{k=0}^1 y_0^{(k)} u_k(t) = y(0) E_\alpha(t^\alpha) + y'(0) \int_0^t E_\alpha(\tau^\alpha) d\tau = 0. \tag{3.38}$$

Which implies that the general form of the solution of the problem (3.35), is:

$$y(t) = 2 - 2E_\alpha(t^\alpha). \tag{3.39}$$

Case of a multidimensional equation

Now, Let the differential equation with fractional-order [36]:

$${}_0^C D_t^\alpha y(t) = Ay(t) + q(t), \tag{3.40}$$

with $0 < \alpha < 1$, $A \in \mathbb{M}_n(\mathbb{R})$, $y \in \mathbb{R}^n$ and $q : [0, h] \longrightarrow \mathbb{R}^n$.

We start by solving the homogeneous problem:

$${}_0^C D_t^\alpha y(t) = Ay(t). \quad (3.41)$$

We distinguish two scenarios.:

• **Scenario 1:** if that the matrix A has simple eigenvalues.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its values, and $v_1, v_2, \dots, v_n$ be the associated eigenvectors.

In this situation, the solution of equation (3.41) is given by:

$$y(t) = \sum_{k=1}^n c_k v_k E_\alpha(\lambda_k t^\alpha), \quad (3.42)$$

where $c_k \in \mathbb{R}, \forall k = 1, \dots, n$.

• **Scenario 2:** Now assume that the matrix A has multiple eigenvalues.

Let λ be its eigenvalue, of multiplicity degree k .

We therefore have:

* If k is equal to the number of eigenvectors associated with λ that are linearly independent, in this situation, the solution of (3.41) is of the form (3.42).

* If m ($m < k$) is equal to the number of eigenvectors associated with λ that are linearly independent, in this situation, the other solutions ($k - m$), which are linearly independent are given by:

$$y^{(i)}(t) = \sum_{j=m}^i u^{(j)} t^{(i-j)\alpha} E_\alpha^{(i-j)}(\lambda t^\alpha), \quad \text{for } i = m + 1, \dots, k, \quad (3.43)$$

where the eigenvectors $u^{(j)}$ represent the solutions of the non-homogeneous linear system:

$$(A - \lambda I)u^{(j+1)} = u^{(j)}. \quad (3.44)$$

Remark 3.1.1. Let $(y_1, y_2, \dots, y_n)^T$ be the solution of the homogeneous problem (3.41), then $(Y_1, Y_2, \dots, Y_n)^T$, the solution of the non-homogeneous problem (3.40), with the initial condition $y(0) = y_0$ is:

$$Y_i(t) = y_i(t) + \int_0^t y_i(t - \tau) q_i(\tau) d\tau \quad \forall i = 1, \dots, n \quad (3.45)$$

Example 3.1.2. (Case 1)

Let the fractional system:

$${}_0^C D_t^\alpha y(t) = Ay(t), \quad y(t) \in \mathbb{R}^2 \text{ et } A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.46)$$

A 's eigenvalues are provided by:

$$\lambda_1 = 1 - \sqrt{2}, \quad \lambda_2 = 1 + \sqrt{2}, \quad (3.47)$$

and the associated eigenvectors are respectively given by:

$$v_1 = (1 - \sqrt{2}, 1)^T, \quad v_2 = (1 + \sqrt{2}, 1)^T. \quad (3.48)$$

Therefore, The system's general solution (3.46) is:

$$y(t) = c_1 \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} E_\alpha((1 - \sqrt{2})t^\alpha) + c_2 \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} E_\alpha((1 + \sqrt{2})t^\alpha). \quad (3.49)$$

Let's choose now

$$y(0) = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix},$$

we then have

$$c_1 = 1 \text{ and } c_2 = -1.$$

Which implies:

$$y(t) = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} E_\alpha((1 - \sqrt{2})t^\alpha) - \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} E_\alpha((1 + \sqrt{2})t^\alpha). \quad (3.50)$$

Example 3.1.3. (Case 2)

Now, consider the system with fractional-order :

$${}_0^C D_t^\alpha y(t) = Ay(t), \quad y \in \mathbb{R}^3 \quad (3.51)$$

where

$$A = \begin{pmatrix} 10 & 2 & -2 \\ 2 & 6 & -2 \\ 4 & 0 & 4 \end{pmatrix}$$

The eigenvalues of A are:

$$\lambda_1 = 8 \text{ (with multiplicity)} 2, \quad \lambda_2 = 4, \quad (3.52)$$

and the associated eigenvectors are respectively given by:

$$v_1 = (1, 0, 1)^T, v_2 = (0, 1, 1)^T. \quad (3.53)$$

Since the number of eigenvectors associated with $\lambda_1 = 8$ is equal to $(m = 1)$ which is strictly less than $k = 2$ (k represents the degree of multiplicity), then the other $(k - m)$ solutions of the system, which are linearly independent, are given by:

$$\begin{aligned} y^{(2)}(t) &= \sum_{j=1}^2 u^{(j)} t^{(2-j)\alpha} E_\alpha^{(2-j)}(\lambda_1 t^\alpha) \\ &= u^{(1)} t^\alpha \dot{E}_\alpha(8t^\alpha) + u^{(2)} E_\alpha(8t^\alpha), \end{aligned} \quad (3.54)$$

where: $u^{(1)} = (1, 0, 1)^T$ satisfying the following condition:

$$(A - \lambda_1 I)u^{(2)} = u^{(1)}. \quad (3.55)$$

Let $u^{(2)} = (a, b, c)^T$. We then have:

$$\begin{cases} 2a + 2b - 2c = 1, \\ 2a - 2b - 2c = 0, \\ 4a - 4c = 1. \end{cases} \quad (3.56)$$

Which gives: $(a, b, c) = (1/4, 1/4, 0)$.

Therefore, The system's general solution(3.51) is :

$$\begin{aligned} y(t) &= c_1(1, 0, 1)^T E_\alpha(8t^\alpha) + c_2(0, 1, 1)^T E_\alpha(4t^\alpha) \\ &+ c_3((1, 0, 1)^T t^\alpha \dot{E}_\alpha(8t^\alpha) + (1/4, 1/4, 0)^T E_\alpha(8t^\alpha)). \end{aligned} \quad (3.57)$$

Under the initial condition $y(0) = (0, 0, 1)$, the constants c_1, c_2 and c_3 are given as:

$$c_1 = 1/2, c_2 = 1/2, \text{ et } c_3 = -2. \quad (3.58)$$

Finally, the solution of the system (3.51) is given by:

$$\begin{aligned} y(t) &= (1/2, 0, 1/2)^T E_\alpha(8t^\alpha) + (0, 1/2, 1/2)^T E_\alpha(4t^\alpha) \\ &+ ((-2, 0, 2)^T t^\alpha \dot{E}_\alpha(8t^\alpha) - (1/2, 1/2, 0)^T E_\alpha(8t^\alpha)). \end{aligned} \quad (3.59)$$

3.1.2 Numerical resolution of non-linear fractional equations

In general, to solve nonlinear differential equations, we use numerical methods, since the analytical solution in this case is usually impossible. There are a number of techniques to solve differential equations with fractional order numerically, including the Adams-Basheforth-Moulton approach, the method of variational iteration, the decomposition method of Adomian, and the method of Grünwald-Letnikov for fractional difference. In this section, we focus on the numerical method of Adams-Basheforth-Moulton, to solve a Caputo-type differential equation with fractional-order. This method is based on a fractional formulation of the classical method of Adams-Basheforth-Moulton. This is the numerical method that we will use throughout this work for the fractional-order simulation dynamical systems.

Generalized Adams-Basheforth-Moulton method:

The Adams-Basheforth-Moulton method is a numerical method introduced by Diethelm and Freed [37], based on the Volterra equation (3.5).

We suppose that y_k is the approximation of $y(t_j)$, for all $j = 1, \dots, k$, in the interval $[0, T]$. To obtain y_{k+1} , we replace the integral in the Volterra equation (3.5), using the formula for the product of quadrature of trapezoids where the nodes t_j for $j = 0, \dots, k+1$, respectively take the function $(t_{k+1} - \cdot)^{\alpha-1}$.

We then obtain the approximation:

$$\int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} g(\tau) d\tau \approx \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \quad (3.60)$$

where

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \phi_{j,k+1} d\tau, \quad (3.61)$$

and

$$\phi_{j,k+1} = \begin{cases} \frac{\tau - t_{j-1}}{t_j - t_{j-1}}, & \text{if } t_{j-1} < \tau < t_j \\ \frac{t_{j+1} - \tau}{t_{j+1} - t_j}, & \text{if } t_j < \tau < t_{j+1} \\ 0, & \text{otherwise.} \end{cases} \quad (3.62)$$

Since $t_j = jh$, for $j = 0, \dots, k+1$, we then get:

$$a_{j,k+1} = \begin{cases} \frac{h^\alpha}{\alpha(\alpha+1)}(k^{\alpha+1} - (k-\alpha)(k+1)^\alpha), & \text{if } j = 0 \\ \frac{h^\alpha}{\alpha(\alpha+1)}((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}), & \text{if } 1 \leq j \leq k \\ \frac{h^\alpha}{\alpha(\alpha+1)}, & \text{if } j = k+1 \end{cases} \quad (3.63)$$

We then find the implicit equation of the Adams-Moulton method at one step:

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}) \right). \quad (3.64)$$

The issue with this last formula is that y_{k+1} that are unknown and appears in both parties of the equation, and because of the nonlinearity of f , it cannot be solved for y_{k+1} in a direct way. To do this, by inserting a preliminary approximate value of y_{k+1} in the right-hand part of this last equality.

y_{k+1}^p represents the preliminary approximation and called the prediction term, is found in the same way, just, by replacing the formula of trapezoid by the formula of rectangle :

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^p) \right). \quad (3.65)$$

Thus, to found y_{k+1}^p , we once again use the one-step method of Adams-Bashforth (similarly for the correction formula), but we substitute the integral with the product rule of rectangles:

$$\int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} g(\tau) d\tau \approx \sum_{j=0}^k b_{j,k+1} g(t_j), \quad (3.66)$$

where:

$$b_{j,k+1} = \frac{h^\alpha}{\alpha} ((k+1-j)^\alpha - (k-j)^\alpha). \quad (3.67)$$

We then have:

$$y_{k+1}^p = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k b_{j,k+1} f(t_j, y_j). \quad (3.68)$$

Finally, the expressions (3.65) and (3.68), with $a_{j,k+1}$ and $b_{j,k+1}$, which are particularly computed from (3.61) and (3.67), respectively, form the method of Adams-Bashforth-Moulton of fractional system.

The name of method is PECE (P(Predict),E(Evaluate),C(Correct), E (Evaluate)) because it starts by computing the predictor using the relation (3.68), then it evaluates $f(t_{k+1}, y_{k+1}^p)$. Using this last evaluation, we compute the corrector via the relation (3.65), and evaluate $f(t_{k+1}, y_{k+1})$ in the end.

3.2 Fractional Chaotic Systems

In autonomous continuous dynamical systems with order less than three, chaos is impossible the model of a chaotic system can be reorganized into three differential equations containing fractional derivatives.

Now, let the following nonlinear system with fractional-order:

$$D^\alpha x = f(x), \quad (3.69)$$

with $x \in \mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ ($0 < \alpha_i < 1$, $i = 1, 2, \dots, n$) and D^α is the derivation operator in the sense of Caputo.

Solving the equation below to calculate the equilibrium points of the system (3.69) :

$$D^\alpha x = 0. \quad (3.70)$$

Remark 3.2.1. In equation (3.69), which describes the dynamics of a nonlinear system of non-integer order, two types of systems will be presented: commensurate systems (or commensurate orders) and non-commensurate systems (or non-commensurate orders).

Definition 3.2.1. If all the orders of derivations α_i , $i = 1, 2, \dots, n$ of the system (3.69) are equal, we say that the system is commensurate. Otherwise, the system is said to be non-commensurate

3.2.1 Differences between systems of fractional and integer order

Let us consider the two systems :

$$\dot{x}(t) = \beta t^{\beta-1}, \quad 0 < \beta < 1, \quad x(0) = x_0 \quad (3.71)$$

$$D^\alpha x(t) = \beta t^{\beta-1}, \quad 0 < \beta < 1, \quad 0 < \alpha < 1 \text{ and } x(0) = x_0 \quad (3.72)$$

The solutions of the systems (3.71) and (3.72), are respectively given by:

$$x(t) = t^\beta + x_0, \quad (3.73)$$

$$x(t) = \frac{\beta \Gamma(\beta) t^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} + x_0. \quad (3.74)$$

It can be easily seen that the solution of the integer system (3.71) is asymptotically stable, for all $0 < \beta < 1$.

However, the solution of the fractional system (3.72) is asymptotically stable, when $0 < \beta < 2 - \alpha$.

This shows that fractional systems have different characteristics than integer systems.

3.2.2 Necessary and sufficient conditions for the stability of the system.

In the theory of stability of linear systems of integer order, a system is considered stable if all the roots of its characteristic polynomial have strictly negative real parts, that is, situated in the complex plane's left half. Whereas, in the situation of linear fractional systems, the definition of stability is strictly different from that of classical systems. Indeed, in fractional systems, we can have roots in the complex plane's right half and be stable.

The following theorem allows us to assert the necessary and sufficient conditions for the stability of fractional-order systems [56].

Theorem 3.2.1. [4] *Let's the non-linear system with fractional-order α :*

$$\begin{cases} D^\alpha Y = f(Y), \\ Y(0) = Y_0. \end{cases} \quad (3.75)$$

where $Y \in \mathbb{R}^n$, $0 < \alpha < 1$, and $f \in \mathbb{R}^n$ is a function that is non-linear and continuous .

1- Suppose that the system (3.75) is commensurate. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ represent the eigenvalues of the Jacobian matrix $\Lambda = \frac{\partial f}{\partial y}$ associated with f at the equilibrium points.

Then, the system (3.75) is considered asymptotically stable if:

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2}, \quad \text{for all } i = 1, 2, \dots, n. \quad (3.76)$$

2- Now suppose that the system (3.75) is non-commensurate, i.e., $\exists i \neq j / \alpha_i \neq \alpha_j$, and let m be the lowest common multiple of the denominators u_i of λ_i , such that:

$$\alpha_i = \frac{u_i}{v_i}, \quad (u_i, v_i) = 1, \quad u_i, v_i \in \mathbb{Z}^+ \quad (3.77)$$

Define $\sigma = \frac{1}{m}$, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the characteristic equation:

$$\det(\text{diag}(\alpha_1^m, \alpha_2^m, \dots, \alpha_n^m) - \Lambda) = 0. \quad (3.78)$$

Then, the non-commensurate system (3.75) is asymptotically stable if:

$$|\arg(\lambda_i)| > \delta \frac{\pi}{2}, \quad \text{for all } i = 1, 2, \dots, n. \quad (3.79)$$

To demonstrate the validity of the preceding theorem, it is necessary to establish the next theorem.

Theorem 3.2.2.

Let consider the function $g(t)$ as a non-constant function and periodic where T its period, suppose that $g(t)$ is m -times differentiable, the fractional derivative of this function ${}^0D_t^\alpha$ cannot be a periodic function with period T (the operator ${}^0D_t^\alpha$ is the operator of fractional derivative of Caputo, Riemann-Liouville or Grünwald-Letnikov)

Proof 3.2.1. 1. Commensurate System

(a) If Λ is diagonalisable, then according to the chapter of fractional differential equation, the solution of (3.75) is given by:

$$Y(t) = \sum_{k=1}^n c_k w_k E_\alpha(\lambda_k t^\alpha), \quad (3.80)$$

by Theorem (2.2.1), its Laplace transform is:

$$Y(s) = \sum_{k=1}^n c_k w_k \frac{s^{\alpha-1}}{s^{\alpha-\lambda_k}}. \quad (3.81)$$

(b) If Λ is not diagonalisable, then according to the part of analytical solution of multidimensional equation, the solution of (3.75) is:

$$Y^{(j,l)}(t) = \sum_{i=0}^{l-1} u^{(j,i+1)} t^{(l-1-i)\alpha} E_\alpha^{(l-1-i)}(\lambda_j t^\alpha) \quad l = 1, \dots, k_j, j = 1, \dots, n, \quad (3.82)$$

with k_j represents the eigenvalue's multiplicity, λ_j . and $\sum_{j=1}^m k_j = n$.

Using the relation 2.2.1 and we apply the Laplace transformation to both parties of equation (3.82), we get:

$$Y^{(j,l)}(s) = \sum_{i=0}^{l-1} u^{(ji+1)} \frac{(l-i-1)! s^{\alpha-1}}{(s^\alpha - \lambda_j)^{(l-1)}}, l = 1, \dots, k_j, j = 1, \dots, m. \quad (3.83)$$

Suppose that every eigenvalue is located in the region $\{\lambda \in \mathbb{C}; |\arg(\lambda)| > \frac{\alpha\pi}{2}\}$. By using equations(3.81) and (3.83), we get:

$$\lim_{y \rightarrow \infty} Y(t) = \lim_{s \rightarrow \infty} sY(s) = 0,$$

Now, we suppose that exists λ_j such that

$$|\arg(\lambda_j)| < \frac{\alpha\pi}{2}$$

then using Theorem 3.2.2, we have:

$$\lim_{y \rightarrow \infty} |E_\alpha(\lambda t^\alpha)| = \infty.$$

Thus from (3.80) and (3.82) $Y(t)$ is unbounded.

2. Non-commensurate system

The next equation is obtained by applying the Laplace transform to both parts of (3.75).

$$(\text{diag}([s^{\alpha_1}, \dots, s^{\alpha_n}]) - \Lambda) Y(s) = (s^{\alpha_1-1} y_1(0), \dots, s^{\alpha_n-1} y_n(0))^T. \quad (3.84)$$

The result of multiplying s on the two parties of (3.84) is:

$$(\text{diag}([s^{\alpha_1}, \dots, s^{\alpha_n}]) - \Lambda) sY(s) = (s^{\alpha_1} y_1(0), \dots, s^{\alpha_n} y_n(0))^T, \quad (3.85)$$

the previous equation does not have one solution $sY(s)$ just when:

$$\det(\text{diag}([s^{\alpha_1}, \dots, s^{\alpha_n}]) - \Lambda) = 0. \quad (3.86)$$

We denote $s = \lambda^{\frac{1}{\delta}} = \lambda^m$ and replacing in (3.86) give us the equation (3.78). If all the roots of the equation (3.86) lie within the complex plane's open left half., $\text{Re}(s) < 0$ (i.e., $|\arg(s)| > \frac{\pi}{2}$, which implies $|\arg(\lambda)| > \delta \frac{\pi}{2}$), then we consider (3.85) in $\text{Re}(s) \leq 0$. In this restricted area, (3.85) has one solution:

$$sX(s) = (sX_1(s), \dots, sX_n(s)).$$

then, we get:

$$\lim_{s \rightarrow 0, \operatorname{Re}(s) \leq 0} sY_i(s) = 0, \quad i = 1, \dots, n.$$

Utilizing the Theorem of final-value 2.2.2, we get:

$$\lim_{t \rightarrow \infty} y_i(t) = \lim_{s \rightarrow 0, \operatorname{Re}(s) \leq 0} sY_i(s) = 0, \quad i = 1, \dots, n.$$

Remark 3.2.2. If the system studied is linear, the stability conditions in the previous theorems remain true, just by replacing the Jacobian matrix of f by its linear part.

Figure 3.1, illustrates the different stability regions of a fractional system, where $0 < \alpha < 1$.

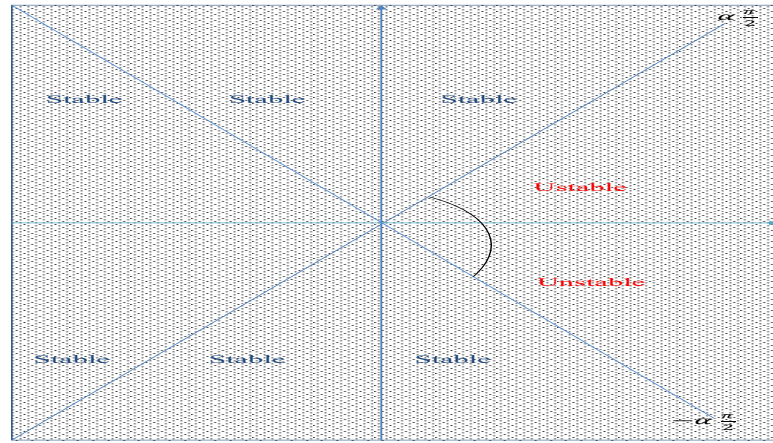


Figure 3.1: The regions of stability for a fractional system, where $0 < \alpha < 1$.

Example 3.2.1. Consider the non-linear fractional-order system :

$$\begin{cases} D^\alpha x = 1 + x^2 y - 4x, \\ D^\alpha y = -x^2 y + 3x, \end{cases} \quad (3.87)$$

where D^α is the Caputo differentiation operator.

Calculate the equilibrium points by solving the next equation:

$$\begin{cases} 1 + x^2 y - 4x = 0, \\ -x^2 y + 3x = 0. \end{cases} \quad (3.88)$$

$P = (1, 3)$ represents the only equilibrium point of (3.87)

At equilibrium point P , the system's Jacobian matrix is:

$$\begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \quad (3.89)$$

It has the next characteristic polynomial:

$$P(\lambda) = \lambda^2 - \lambda + 1. \quad (3.90)$$

The system's eigenvalues related to point P are provided by:

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i. \quad (3.91)$$

When $\alpha = 0.65$, we can easily see that:

$$|\arg(\lambda_{1,2})| = 1.047 > \frac{\alpha\pi}{2} = 1.021. \quad (3.92)$$

Using the Theorem 3.2.1, shows the equilibrium point stability P . However, in the integer case, the equilibrium point P is unstable.

3.2.3 Necessary condition for chaos

Looking at the commensurate system (3.75), it is evident that the equilibrium points are found by solving the equation $f(y) = 0$. According to Theorem 3.2.1, if all the eigenvalues of the Jacobian matrix $(\frac{\partial f}{\partial y})$ for these points satisfy the condition in (3.76), then these points are considered locally asymptotically stable.

In the theory of three-dimensional dynamical systems, an equilibrium is called a saddle point, when linearized system that is corresponding to an equilibrium has at least one eigenvalue in stable region and the others are in unstable region. In addition, if and only if, the corresponding linearized system has one stable eigenvalue and the rest are unstable, this point is referred to as saddle point of index 1 and is referred to as saddle point of index 2 if and only if, two eigenvalue is unstable region and one eigenvalue in stable region.

For chaotic system, it is commonly know that "scrolls" are product only at index 2 saddle point, on the other hand index 1 saddle point is responsible for joining these scrolls.

Assume that we have a three-dimension chaotic system, has an attractor of a single scroll. Then this system has a index 2 saddle point, surrounded by this attractor. Let λ an eigenvalue of index 2 saddle point and λ if λ satisfies the condition:

$$\alpha > \frac{2}{\pi} \arctan\left(\left|\frac{\Re(\lambda)}{\Im(\lambda)}\right|\right). \quad (3.93)$$

then, the fractional system (3.69) is able to have a chaotic behavior.

3.2.4 Direct fractional Lyapunov method

The concept of the direct Lyapunov method consists in finding a Lyapunov function associated with a nonlinear problem. When there is this function, the system is stable. This method is difficult to implement, however it is of a much more general scope. Note that this method allows us to give a sufficient condition of stability, that is, we can demonstrate the stability of a system, even in the face of the impossibility of finding a Lyapunov function, since we are unable to identify such a Lyapunov function using any generic rule. However, in physics problems, energy is often a good candidate. Our extension in this part, is the direct Lyapunov method of fractional systems to study the asymptotic stability of solutions of the following system:

$$D^\alpha x(t) = f(x(t)), \quad (3.94)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents a nonlinear function, $x \in \mathbb{R}^n$ is the state vector and $\alpha \in (0, 1)$.

To achieve this objective, we recall some classical results of stability in the sense of Lyapunov, on which our work described in the last chapters of this work is based.

Theorem 3.2.3. [40] *If there is a positive function of Lyapunov $V(x)$, and the Caputo fractional derivative of this function is strictly negative, i. e. , $(D^\alpha(V(x)) < 0, \forall t \geq t_0)$, then the solution of (3.94) is asymptotically stable.*

3.2.5 Lyapunov candidate functions for stability

This section presents a new property for fractional derivatives with Caputo derivative, when $0 < \alpha < 1$, this enables the expansion of the direct Lyapunov method of fractional order to develop a simple candidate Lyapunov function for numerous fractional order systems and, therefore, the evidence of their stability.

Lemme 3.2.1. [40] Let $h \in \mathbb{R}$ be a Caputo-differentiable function. $\forall t \geq t_0$, we have:

$$\frac{1}{2}D^\alpha h^2(t) \leq h(t)D^\alpha h(t), \quad \alpha \in (0, 1). \quad (3.95)$$

Proof 3.2.2. By definition, we have:

$$D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{\dot{h}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (3.96)$$

i.e.,:

$$\frac{1}{2}D^\alpha h^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{h(\tau)\dot{h}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (3.97)$$

Thus:

$$h(t)D^\alpha h(t) - \frac{1}{2}D^\alpha h^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{[h(t) - h(\tau)]\dot{h}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (3.98)$$

Making the modification:

$$y(\tau) = h(t) - h(\tau),$$

The equation (3.98) is able to be described as :

$$h(t)D^\alpha h(t) - \frac{1}{2}D^\alpha h^2(t) = \frac{-1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{y(\tau)\dot{y}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (3.99)$$

Let's integrate by part, we put:

$$\begin{cases} du = y(\tau)\dot{y}(\tau)d\tau, \\ v = \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}, \end{cases} \quad (3.100)$$

then:

$$h(t)D^\alpha h(t) - \frac{1}{2}D^\alpha h^2(t) = \lim_{t \rightarrow \tau} \frac{-y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} + \frac{y^2(t_0)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} + \frac{1}{2\Gamma(1-\alpha)} \int_{t_0}^t \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau. \quad (3.101)$$

Applying the rule of hospital , we obtain:

$$\lim_{t \rightarrow \tau} \frac{-y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} = \lim_{t \rightarrow \tau} \frac{2y(\tau)\dot{y}(\tau)}{2\alpha\Gamma(1-\alpha)(t-\tau)^{\alpha-1}} = 0. \quad (3.102)$$

Finally:

$$h(t)D^\alpha h(t) - \frac{1}{2}D^\alpha h^2(t) = \frac{y^2(t_0)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} + \frac{1}{2\Gamma(1-\alpha)} \int_{t_0}^t \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau \geq 0, \text{ for all } t \geq t_0. \quad (3.103)$$

Remark 3.2.3. The previous lemma is true, if $h(t) \in \mathbb{R}^n$, and in this case:

$$\frac{1}{2}D^\alpha h^T(t)h(t) \leq h^T(t)D^\alpha h(t), \text{ for all } \alpha \in (0, 1) \quad (3.104)$$

Corollary 3.2.1. *Let's consider the following system with fractional-order :*

$$D^\alpha h(t) = f(h(t)), \quad (3.105)$$

with $\alpha \in (0, 1)$. Let $h = 0$, the equilibrium of this system if the condition.

$$h(t)f(h(t)) \leq 0, \forall h, \text{ (resp. } h(t)f(h(t)) < 0, \text{ for all } h \neq 0, \text{)} \quad (3.106)$$

satisfied, then the equilibrium 0 is a stable point (or asymptotically stable)

Proof. Let's the Lyapounov function:

$$V(h(t)) = \frac{1}{2}h^2(t) \quad (3.107)$$

using the previous lemma, we have:

$$D^\alpha V(h(t)) \leq h(t)D^\alpha h(t) = h(t)f(h(t)) \leq 0, \forall h \text{ (resp. } < 0, \forall h \neq 0), \quad (3.108)$$

then, using the Theorem 3.2.3, 0 is a stable point (or asymptotically stable). $\square$

Example 3.2.2. *Examine the model of satellite chaotic system with fractional-order [41], which can be given by:*

$$\begin{cases} D^\alpha x = \frac{1}{3}yz - 0.4x + \frac{1}{\sqrt{6}}z, \\ D^\alpha y = -xz + 0.175y, \\ D^\alpha z = xy - \sqrt{6}x - 0.4z. \end{cases} \quad (3.109)$$

where:

* x, y and z represent the system's state variables.

* $0 < \alpha < 1$.

* D^α represents the operator of Caputo fractional derivative with order α .

The attractors of chaotic system (3.109) are shown in Figure 3.2 and the temporal evolution of the states x, y and z are shown in Figure 3.3.

To analyse the asymptotically stability of the system, we consider the controlled chaotic system as follows

$$\begin{cases} D^\alpha x = \frac{1}{3}yz - 0.4x + \frac{1}{\sqrt{6}}z, \\ D^\alpha y = -xz + 0.175y - ky, \\ D^\alpha z = xy - \sqrt{6}x - 0.4z, \end{cases} \quad (3.110)$$

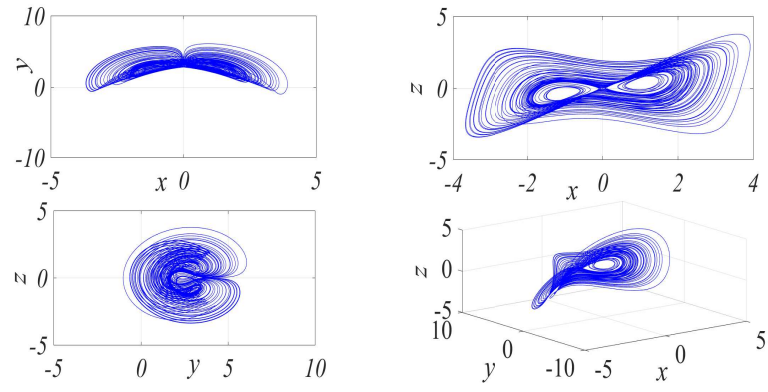


Figure 3.2: Attractors of the chaotic system (3.109).

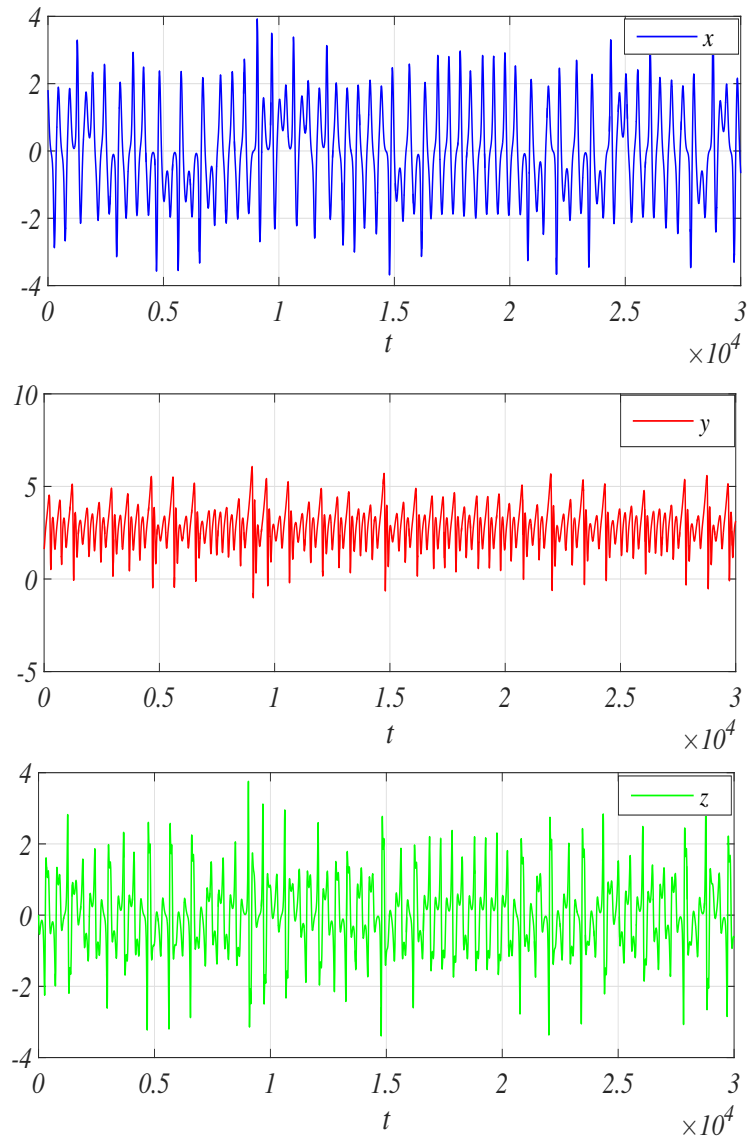


Figure 3.3: Temporel evolution of the states x , y and z .

where k is positive feedback control gains.

If the feedback control gains $k > 0.175$, then the controlled chaotic system (6.1) is asymptotically stable.

Define the Lyapunov functional as

$$V(x, y, z) = \frac{1}{2} (6x^2 + 3y^2 + z^2).$$

By using Lemma 3.2.1, the fractional-order derivative of V is given by

$$\begin{aligned} D^\alpha V(x, y, z) &\leq 6xD^\alpha x + 3yD^\alpha y + zD^\alpha z \\ &= 2xyz - 2.4x^2 + \frac{6}{\sqrt{6}}xz + \\ &\quad -3xyz - 3(k - 0.175)y^2 + \\ &\quad +xyz - \sqrt{6}xz - 0.4z^2 \\ &\leq -2.4x^2 - 3(k - 0.175)y^2 - 0.4z^2. \end{aligned}$$

If the feedback control gains $k > 0.175$, then $D^\alpha V(x, y, z) < 0$ for $(x, y, z) \neq (0, 0, 0)$.

Thus, the system (6.1) asymptotically converges to zero.

For the numerical simulations, we use the method of Adams-Bashforth-Moulton [37] to solve this system.

We assume that the starting conditions of the system are employed as

$$x(0) = 0.1, \quad y(0) = 0.1 \text{ and } z(0) = 0.2$$

To ensure the condition of stability, the control input is chosen as $k = 0.5$, and the fractional-order is selected as $\alpha = 0.98$.

The controlled chaotic system's time-history is displayed in Figure 3.4.

It is evident that every solution to this system approaches the origin values, confirming the asymptotic stability of the controlled chaotic system (3.110).

Example 3.2.3. Consider the system;

$$\begin{cases} D^\alpha x(t) = -x(t) + y^3(t), \\ D^\alpha y(t) = -x(t) - y(t), \end{cases} \quad (3.111)$$

Define the Lyapunov function as:

$$V(x(t), y(t)) = \frac{1}{2}x^2(t) + \frac{1}{4}y^4(t). \quad (3.112)$$

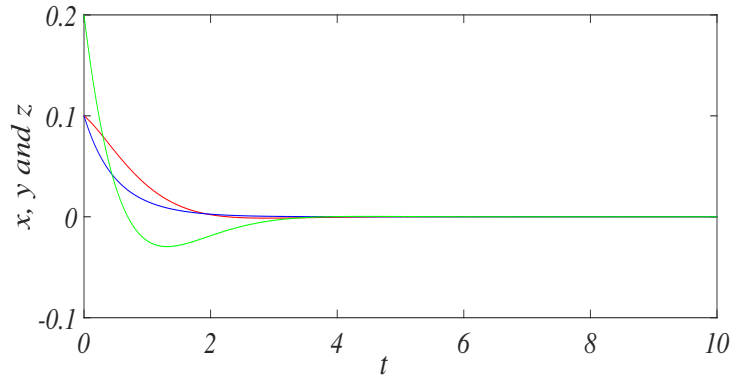


Figure 3.4: Time-history of the controlled chaotic system (3.110).

It is evident that the V is a function that is positive definite.

On the other hand, using Lemma 3.2.1, the derivative of fractional-order of V is:

$$\begin{aligned}
 D^\alpha V(x(t), y(t)) &= \frac{1}{2}D^\alpha x^2(t) + \frac{1}{4}D^\alpha y^4(t) \\
 &\leq x(t)D^\alpha x(t) + \frac{1}{2}y^2(t)D^\alpha y^2(t) \\
 &\leq x(t)D^\alpha x(t) + y^3(t)D^\alpha y(t) \\
 &\leq -x^2(t) - y^4(t) \\
 &< 0, \text{ for } (x, y) \neq (0, 0)
 \end{aligned} \tag{3.113}$$

Thus, the system (3.111) asymptotically converges to zero.

Synchronization of chaotic system

The word "synchronization" comes from a Greek root that means "to share the common moment." Until now, the term "synchronization" has been used colloquially to refer to the agreement or connection in time of different activities. There has been a lot of research done on the study of synchronization experiences in the dynamical systems evolution. It began in the 17th century, when Huygens discovered that two pendulum clocks that were hanging at the same beam, but they were very weakly related might be made to synchronize in phase. The coordinated lighting of fireflies is another early example.

Recently, chaotic systems have become the focus of synchronization search, when there is a shared behavior between two, three or more chaotic systems, whether they are equivalent or not, due to a coupling or forcing of some part of their motion (either noisy or periodic), this is known as synchronization of chaos. Its concept is two chaotic systems may develop on different attractors, but when they are linked, they begin on different attractors and eventually end up following a similar path, since the solutions of chaotic systems with local initial conditions diverge quickly.

A combination synchronization strategy between three chaotic systems was the subject of two papers published in 2011 and 2012 [42, 43]. It was initially demonstrated by these authors that it was possible to synchronize the response system's state variables with the total of the state variables of two driving systems.

This chapter discusses the phenomenon of synchronization of chaotic dynamical systems. First, we will discuss the types of synchronization and combination synchronization, and we present an application of compound combination synchronization of

one fractional-order system and three integer-order systems.

4.1 Synchronization Techniques

Conventional synchronization techniques typically rely on using circuits that are identical. Let us consider two identical chaotic systems that oscillate entirely on their own. If they are permitted to exchange energy, a process known as "coupling," the two systems will eventually adopt a common behavior: they synchronize. Chaotic systems can couple either both ways (bidirectional coupling) or in just one direction (unidirectional coupling).

4.1.1 Unidirectional coupling

This kind of arrangement is called "master-slave" or "drive-response" and occurs when energy is transferred from one system to another after it has been divided into two smaller systems, one of which leads and the other follows.

Consider two identical systems as follows:

$$D^\alpha X_1(t) = h(X_1, t), \quad (4.1)$$

and

$$D^\alpha X_2(t) = h(X_2, t), \quad (4.2)$$

the two equations (4.1) and (4.2) are said to be unidirectionally coupled, if the second equation changes and two new equations are created:

$$D^\alpha X_1(t) = h(X_1, t), \quad (4.3)$$

and

$$D^\alpha X_2(t) = \begin{cases} h(X_2, t) & \text{if } X_1 = X_2, \\ h(X_1, X_2, t) & \text{otherwise.} \end{cases} \quad (4.4)$$

4.1.2 Bidirectional coupling

When two subsystems are connected to one another in some manner, their trajectories are impacted by one another's actions. This is known as bidirectional coupling.

$$D^\alpha X_1(t) = h_1(X_1, t), \quad (4.5)$$

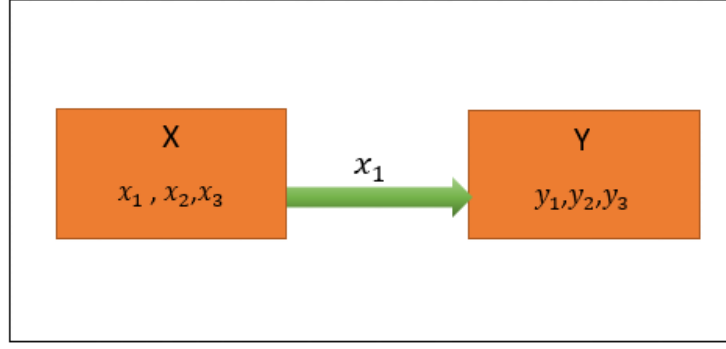


Figure 4.1: Unidirectional coupling

and

$$D^\alpha X_2(t) = h_2(X_2, t), \quad (4.6)$$

If both systems (4.5) and (4.6) can be rewritten in the following ways:

$$D^\alpha X_1(t) = g_1(X_1, X_1, X_2, t), \quad (4.7)$$

and

$$D^\alpha X_2(t) = g_2(X_2, X_1, X_2, t), \quad (4.8)$$

they are considered coupled, where g_1 and g_2 are non-linear functions, the system variable is represented by the first variable in each equation, and the coupling effect's output is represented by the second and third variables. The following example demonstrates

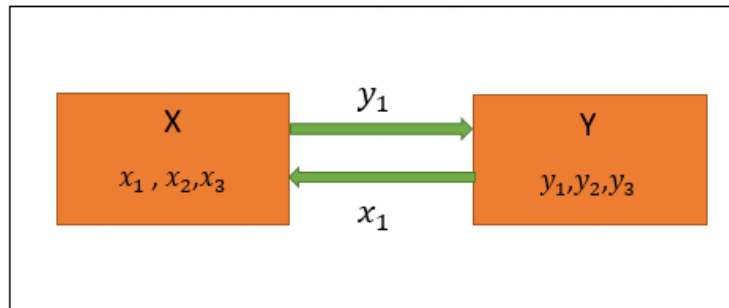


Figure 4.2: Bidirectional coupling

the difference between unidirectional and bidirectional coupling synchronization, in an easily understood way.

Example 4.1.1. *Let two identical fractional chaotic systems:*

$$D^\alpha x = f(x), \quad (4.9)$$

and

$$D^\alpha y = g(y), \quad (4.10)$$

with $x = (x_1, x_2, x_3)^T$ are the state vector of the system(4.9), $y = (y_1, y_2, y_3)^T$ are the state vector of the system(4.10), $f = (f_1, f_2, f_3)^T$ and $g = (g_1, g_2, g_3)^T$. are continuous vector function of the system (4.9)and (4.10) respectively

The coupling between these systems can be given , using the equations:

$$\begin{cases} D^\alpha x_1 = f_1(x) + k_1(y_1 - x_1), \\ D^\alpha x_2 = f_2(x) + k_2(y_1 - x_1), \\ D^\alpha x_3 = f_3(x) + k_3(y_1 - x_1), \end{cases} \quad (4.11)$$

and

$$\begin{cases} D^\alpha y_1 = g_1(x) + r_1(x_1 - y_1), \\ D^\alpha y_2 = g_2(x) + r_2(x_1 - y_1), \\ D^\alpha y_3 = g_3(x) + r_3(x_1 - y_1), \end{cases} \quad (4.12)$$

with $k_i, r_i, i = 1, 2, 3$ are constants of coupling.

We have two cases:

1. $k_i = 0 \forall i = 1, 2, 3$: the system (4.9) 's state affects system (4.10), whereas system (4.10) has no effect on system (4.9), which means that there is a unidimensional coupling between the two systems. Then, system (4.9) is the transmitter or master, and system (4.10) is the slave or receiver.
2. if $\exists i$ with $k_i \neq 0$ and $r_i \neq 0 \forall i = 1, 2, 3$, a bidirectional coupling is created between the two systems, meaning that each influences the other, and also vice versa.

4.2 Synchronization types

Consider the drive system:

$$D^\alpha x_1 = f_1(x_1), \quad (4.13)$$

and the controlled response system:

$$D^\alpha x_2 = f_2(x_2) + u, \quad (4.14)$$

where x_1 and $x_2 \in \mathbb{R}^n$ are state vectors, f_1 and f_2 are non linear functions and u is a controller to be determined.

Let $x_1(0)$ and $x_2(0)$ be a initial conditions of systems (4.13) and (4.14), respectively. We have the following definitions.

Definition 4.2.1. (Complete Synchronization)

The two systems (4.13) and (4.14) are said to be complete synchronization, if their states coincide over time. It holds, if the condition

$$\lim_{t \rightarrow \infty} \|x_1(t) - x_2(t)\| = 0.$$

is satisfied

Remark 4.2.1. If $f_1 = f_2$, this synchronization called identical complete synchronization.

Definition 4.2.2. (Anti-synchronization)

If there is a controller $u = (u_1, u_2, \dots, u_n)^T$, such that:

$$\lim_{t \rightarrow \infty} \|x_1(t) + x_2(t)\| = 0,$$

then, the systems (4.13) and (4.14) are achieved anti-synchronization, for all initial conditions $x_1(0), x_2(0)$.

Definition 4.2.3. (Generalized synchronization)

The two systems (4.13) and (4.14) are synchronized in the generalized sense, if exists a controller $u = (u_1, u_2, \dots, u_n)^T$ and a transformation M such that:

$$\lim_{t \rightarrow \infty} \|x_1(t) - M(x_2)(t)\| = 0,$$

there is no consideration of the initial conditions in this particular case.

Definition 4.2.4. (Projective synchronization)

We say that exists is a projective synchronization of the systems (4.13) and (4.14), if there is an effective control $u(x_1; x_2)$, where, for all initial conditions $x_1(0)$ and $x_2(0)$ of the systems (4.13) and (4.14), the following equation :

$$\lim_{t \rightarrow \infty} \|x_1(t) - \theta x_2(t)\| = 0,$$

is satisfied, where θ is a non-zero constant known as the scale factor.

Definition 4.2.5. (Modified Projective synchronization)

If there is an effective control $u(x_1; x_2)$ and if the following expression is satisfied:

$$\lim_{t \rightarrow \infty} \|\Theta x_1(t) - x_2(t)\| = 0,$$

where Θ is a non-zero constant matrix known as scale matrix, then there is a modified projective synchronization between the systems (4.13) and (4.14).

Definition 4.2.6. (Q – S synchronization)

If there is an effective control $u(x_1; x_2)$ and if the following expression is satisfied:

$$\lim_{t \rightarrow \infty} \|Qx_1(t) - Sx_2(t)\| = 0,$$

where Q and S are non-zero constant matrices, then there is a $Q - S$ synchronization between the systems (4.13) and (4.14).

4.3 Combination synchronization

All the previous methods focus on the usual synchronization between two chaotic systems, that is a very special situation of combination synchronization. In the year 2011, Runzi et al. [16] introduced a novel method for synchronizing two drive systems and one response system, known as "combination synchronization," in which the combination of trajectories of two drive systems and one response system remain similar. Combination synchronization can significantly improve secure information's resistance to attacks and decoding. Many types of kind of this synchronization has been developed in recent years, such as dual combination synchronization [44, 45], generalized combination synchronization [46], projective combination synchronization [47], and also combination-combination synchronization [48].

Consider the drive systems:

$$D^\alpha x_3 = f_3(x_3), \tag{4.15}$$

$$D^\alpha x_4 = f_4(x_4), \tag{4.16}$$

and other controlled response system as:

$$D^\alpha x_5 = f_5(x_5) + v, \tag{4.17}$$

where x_3, x_4 and $x_5 \in \mathbb{R}^n$ is state vectors, f_3, f_4 and f_5 are non linear functions and v is an other controller to be determined.

We have.

Definition 4.3.1. (Combination snchronization)

If there is an effective control $v(x_3, x_4, x_5)$ and if the following expression is satisfied:

$$\lim_{t \rightarrow \infty} \|Ax_3(t) + Bx_4(t) + Cx_5(t)\| = 0,$$

with A, B and C are non-zero constant matrices, then exists a combination synchronization between the systems (4.15), (4.16) and (4.17).

Definition 4.3.2. (Compound combination snchronization)

If there is an effective control $u(x_1, x_2, x_3, x_4)$ and if the following expression is satisfied:

$$\lim_{t \rightarrow \infty} \|Ax_2(t) + Bx_1(t)(Cx_3(t) + Dx_4(t))\| = 0,$$

with A, B, C and D are non-zero diagonal constant matrices, then there is a compound combination synchronization between the systems (4.13), (4.15), (4.16) and (4.14).

Definition 4.3.3. (Combination-combination snchronization)

If there is an effective control $u(x_1, x_3, x_2, x_5)$ and if the following expression is satisfied:

$$\lim_{t \rightarrow \infty} \|Ax_2(t) + Bx_5(t) + Cx_1(t) + Dx_3(t)\| = 0,$$

with A, B, C and D are non-zero constant matrices, then there is a combination- combination synchronization between the systems (4.13)-(4.15) and (4.14)-(4.17).

Compound combination synchronization (CCS) between four unified chaotic systems

In the present chapter, our goal is to examine the problem of compound combination synchronization, we study this problem between four distinct non integer-order systems that are chaotic and similar. Utilizing the theory of stability of linear systems, Laplace transformation and a novel suitable control law is proposed to confirm that this type of synchronization is achieved. This control technique is applied for realised compound combination synchronization between four united chaotic systems that are identical. To illustrate the efficacy of the suggested approach, simulations are provided.

5.1 CCS's general control method

This section designs compound combination synchronization of several fractional-order chaotic systems.

The scaling drive system is:

$$\dot{x} = f(x), \quad (5.1)$$

the first base drive system is:

$$\dot{y} = g(y), \quad (5.2)$$

the second base drive system is:

$$\dot{z} = h(z), \quad (5.3)$$

where:

- * $x \in \mathbb{R}^n$ is the state vector associated with the first drive system (5.1).
- * $y \in \mathbb{R}^n$ demotes the drive system's (5.2) state vector.
- * $z \in \mathbb{R}^n$ demotes the state vector associated with the first drive system (5.3) .
- * f, g and $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ represent vector functions that are continuous.

The response system is defined as:

$$D^p w = R(w) + u, \quad (5.4)$$

with:

- * $p \in \mathbb{Q}$ such that: $0 < p < 1$.
- * D^p represents the Caputo operator of the fractional order p .
- * $w \in \mathbb{R}^n$ represents the response system's state vector(5.4).
- * $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is function that is continuous .
- * $u \in \mathbb{R}^n$ is an active controller.

The concept of compound combination synchronization between (5.1)-(5.3) and (5.4) is expressed as:

Definition 5.1.1. *The drive systems (5.1)-(5.3) and the response system (5.4) are achieved CCS, if there are four diagonal matrices $V = \text{diag}(v_1, v_2, \dots, v_n)$, $N = \text{diag}(n_1, n_2, \dots, n_n)$, $\Psi = \text{diag}(\psi_1, \psi_2, \dots, \psi_n)$ and $Q = \text{diag}(q_1, q_2, \dots, q_n)$ and a controller u such that*

$$\lim_{t \rightarrow +\infty} \|(QW + VX(\Psi Z - NY))(t)\| = 0.$$

Here X, Y, Z and W are diagonal matrices defined as:

$$\begin{aligned} X &= \text{diag}(x_1, x_2, \dots, x_n), \\ Y &= \text{diag}(y_1, y_2, \dots, y_n), \\ Z &= \text{diag}(z_1, z_2, \dots, z_n), \\ W &= \text{diag}(w_1, w_2, \dots, w_n). \end{aligned} \quad (5.5)$$

The state error is :

$$e = QW + VX(\Psi Z - NY). \quad (5.6)$$

We have:

$$\begin{aligned} \dot{e} &= Q\dot{W} + V\dot{X}(\Psi Z - NY) + VX(\Psi\dot{Z} - N\dot{Y}) \\ &= Q\dot{W} + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} F(x) &= \text{diag}(f_1(x), f_2(x), \dots, f_n(x)), \\ G(y) &= \text{diag}(g_1(y), g_2(y), \dots, g_n(y)), \\ H(z) &= \text{diag}(h_1(z), h_2(z), \dots, h_n(z)). \end{aligned} \quad (5.8)$$

Here, the current goal is to identify a suitable controller u , where the error of synchronization (5.7) tends to zero, which indicates that the compound combination synchronization between the systems (5.1)-(5.3) and (5.4) is achieved. To verify the synchronization strategy, the linear part of the error system is controlled.

To achieve this goal, a novel control law technique is proposed:

Assume that function of the control u is:

$$u = -R(w) + I^{1-p} [R(w) + \kappa], \quad (5.9)$$

with κ is:

$$\kappa = -Q^{-1}\zeta, \quad (5.10)$$

Q^{-1} represents the inverse matrix of matrix Q .

And

$$\zeta = -(P + M)e + QR(w) + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)), \quad (5.11)$$

and

* $R(w) = \text{diag}(r_1(w), r_2(w), \dots, r_n(w))$.

* P denotes the linear component of the system under study .

* M represents an unidentified control matrix that has to be found..

Then, we get:

Theorem 5.1.1. *Using the active controller (5.9), the drive systems (5.1)-(5.3) and the response system (6.3), will be achieve compound combination synchronization.*

Proof. Replacing the controller (5.9) into Equ. (5.4), we get:

$$\begin{aligned} D^p w &= R(w) - R(w) + I^{1-p} [R(w) + \kappa] \\ &= I^{1-p} [R(w) + \kappa]. \end{aligned} \quad (5.12)$$

Introducing the Laplace transform (2.33) into (5.12), with $F(s) = L(w)$, we obtain:

$$s^p F(s) - s^{p-1} w(0) = s^{p-1} L(R(w) + \zeta). \quad (5.13)$$

Multiplying both sides of Equ. (5.13) by s^{1-p} , we get:

$$sF(s) - w(0) = L(R(w) + \zeta). \quad (5.14)$$

Introducing the inverse of Laplace transformation to Equ.(5.14), we obtain:

$$\dot{w} = R(w) + \kappa. \quad (5.15)$$

Substituting Equ. (5.15) into (5.7) and using the function ζ that is described by (5.10), we get:

$$\begin{aligned} \dot{e} &= Q\dot{w} + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \\ &= Q(R(w) + \kappa) + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \\ &= QR(w) + Q\kappa + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)), \\ &= QR(w) + Q(-Q^{-1}\zeta) + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \\ &= QR(w) - \left[-(P + M)e + QK(w) + VF(x)(\Psi Z - NY) \right. \\ &\quad \left. + NX(\Psi H(z) - NG(y)) \right] + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \\ &= QR(w) + (P + M)e - QR(w) - VF(x)(\Psi Z - NY) - VX(\Psi H(z) - NG(y)) \\ &\quad + VF(x)(\Psi Z - NY) + VX(\Psi H(z) - NG(y)) \\ &= (P + M)e. \end{aligned} \quad (5.16)$$

According to the dynamical linear systems stability theorem, if we chose the matrix M , so that the eigenvalues of the matrix $(P + M)$ has strictly negative real parts, it might be remarked that the error system (5.16) is asymptotically approaches to zero, that indicate the response systems (5.4) will provide the compound combination synchronization with the three drive systems (5.1)-(5.3). $\square$

5.2 Numerical simulation results

To confirm that the specified synchronization strategy is successful, three unified chaotic systems with integer-order are selected like the drive systems and for the response , one united chaotic system with fractional-order is taken.

The united chaotic systems with integer-order(which are Lorenz, Chen and Lu systems) [49] are:

$$\begin{cases} \dot{x}_1 = (25\delta + 10)(x_2 - x_1), \\ \dot{x}_2 = (-35\delta + 28)x_1 - x_1x_3 + (29\delta - 1)x_2, \\ \dot{x}_3 = x_1x_2 - (\frac{\delta + 8}{3})x_3. \end{cases} \quad (5.17)$$

$$\begin{cases} \dot{y}_1 = (25\delta + 10)(y_2 - y_1), \\ \dot{y}_2 = (-35\delta + 28)y_1 - y_1y_3 + (29\delta - 1)y_2, \\ \dot{y}_3 = y_1y_2 - (\frac{\delta + 8}{3})y_3, \end{cases} \quad (5.18)$$

and

$$\begin{cases} \dot{z}_1 = (25\delta + 10)(z_2 - z_1) \\ \dot{z}_2 = (-35\delta + 28)z_1 - z_1z_3 + (29\delta - 1)z_2, \\ \dot{z}_3 = z_1z_2 - (\frac{\delta + 8}{3})z_3. \end{cases} \quad (5.19)$$

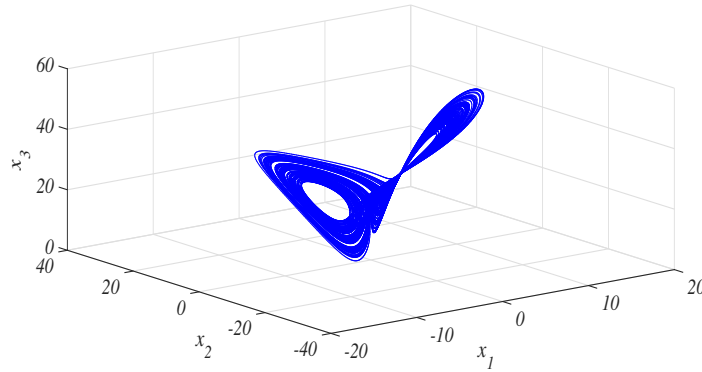


Figure 5.1: The Lorenz attractor system (5.17), with $p = 1$ and $\delta = 0$.

The united chaotic system with fractional order and controlled is :

$$\begin{cases} D_1^p w_1 = (25\delta + 10)(w_2 - w_1) + u_1, \\ D_1^p w_2 = (-35\delta + 28)w_1 - w_1w_3 + (29\delta - 1)w_2 + u_2, \\ D_1^p w_3 = w_1w_2 - (\frac{\delta + 8}{3})w_3 + u_3, \end{cases} \quad (5.20)$$

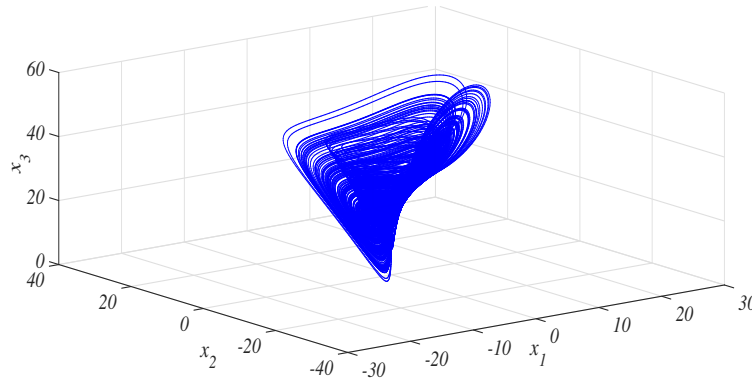


Figure 5.2: The Lü attractor system (5.17), with $p = 1$ and $\delta = 0.8$.

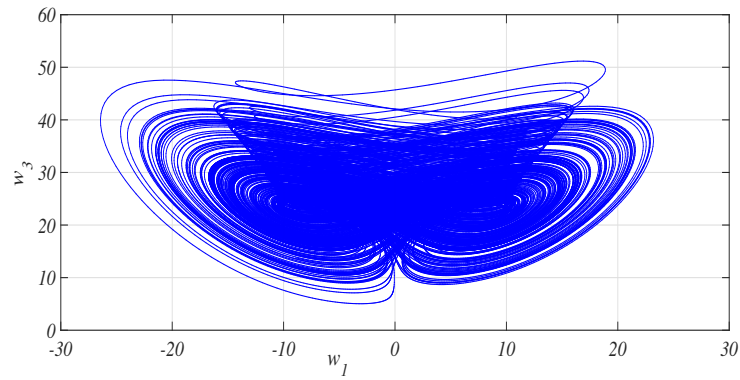


Figure 5.3: The attractor of Chen system (5.20), with $\delta = 1$ and $p = 0.98$.

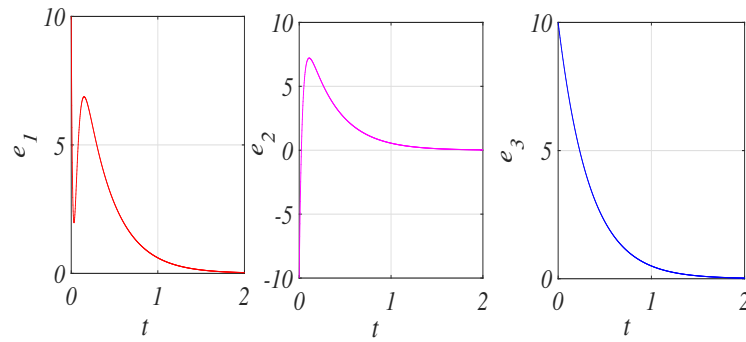


Figure 5.4: The evolution of synchronization errors(5.22),with $\delta = 1$ over time

with:

*For the drive systems the state variables are: x_i, y_i and z_i , and the state variables of the response system are represented by w_i with $i = 1, 2, 3$.

* $0 \leq \delta \leq 1$.

* u_1, u_2, u_3 indicate the active controllers that need to be created..

V, Ψ, N and Q are :

$$V = \text{diag}(v_1, v_2, v_3)$$

$$N = \text{diag}(n_1, n_2, n_3)$$

$$\Psi = \text{diag}(\psi_1, \psi_2, \psi_3)$$

$$Q = \text{diag}(q_1, q_2, q_3)$$

In view of CCS control method suggested in the theoretical part, the error states can be defined like :

$$\begin{cases} e_1 = q_1 w_1 + v_1 x_1 (\gamma_1 z_1 - n_1 y_1), \\ e_2 = q_2 w_2 + v_2 x_2 (\psi_2 z_2 - n_2 y_2), \\ e_3 = q_3 w_3 + v_3 x_3 (\psi_3 z_3 - n_3 y_3). \end{cases}$$

The error system can be derived as follows:

$$\begin{cases} \dot{e}_1 = q_1 \dot{w}_1 + v_1 \dot{x}_1 (\psi_1 z_1 - n_1 y_1) + v_1 x_1 (\psi_1 \dot{z}_1 - n_1 \dot{y}_1), \\ \dot{e}_2 = q_2 \dot{w}_2 + v_2 \dot{x}_2 (\psi_2 z_2 - n_2 y_2) + v_2 x_2 (\psi_2 \dot{z}_2 - n_2 \dot{y}_2), \\ \dot{e}_3 = q_3 \dot{w}_3 + v_3 \dot{x}_3 (\psi_3 z_3 - n_3 y_3) + v_3 x_3 (\psi_3 \dot{z}_3 - n_3 \dot{y}_3). \end{cases} \quad (5.21)$$

Using the Laplace transform and the controller (5.9), we get:

$$\begin{cases} \dot{e}_1 = q_1(R_1(w) + \kappa_1) + v_1 \dot{x}_1 (\psi_1 z_1 - n_1 y_1) + v_1 x_1 (\psi_1 \dot{z}_1 - n_1 \dot{y}_1), \\ \dot{e}_2 = q_2(R_2(w) + \kappa_2) + v_2 \dot{x}_2 (\psi_2 z_2 - n_2 y_2) + v_2 x_2 (\psi_2 \dot{z}_2 - n_2 \dot{y}_2), \\ \dot{e}_3 = q_3(R_3(w) + \kappa_3) + v_3 \dot{x}_3 (\psi_3 z_3 - n_3 y_3) + v_3 x_3 (\psi_3 \dot{z}_3 - n_3 \dot{y}_3). \end{cases}$$

The suggested system's linear element is :

$$P = \begin{pmatrix} -(25\delta + 10) & (25\delta + 10) & 0 \\ (-35\delta + 28) & (29\delta - 1) & 0 \\ 0 & 0 & -(\frac{\delta + 8}{3}) \end{pmatrix}$$

To ensure the stability condition, we choose M as as:

$$M = \begin{pmatrix} 0 & 0 & 0 \\ -(-35\delta + 28) & -58\delta & (29\delta - 1) \\ 0 & 0 & -(\frac{\delta + 8}{3}) \end{pmatrix}$$

then:

$$(P + M)e = \begin{pmatrix} -(25\delta + 10) & (25\delta + 10) & 0 \\ 0 & -(29\delta + 1) & (29\delta - 1) \\ 0 & 0 & -2\left(\frac{\delta + 8}{3}\right) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

The matrices Q and Q^{-1} are :

$$Q = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix}.$$

and

$$Q^{-1} = \begin{pmatrix} \frac{1}{q_1} & 0 & 0 \\ 0 & \frac{1}{q_2} & 0 \\ 0 & 0 & \frac{1}{q_3} \end{pmatrix}.$$

Now we can calculate the vector v :

$$\begin{cases} \kappa_1 = -\frac{1}{q_1} \left[-\left[(25\delta + 10)(e_1 - e_2) \right] + q_1 R_1(w) + n_1 \dot{x}_1 (\psi_1 z_1 - n_1 y_1) + v_1 x_1 (\psi_1 \dot{z}_1 - n_1 \dot{y}_1) \right], \\ \kappa_2 = -\frac{1}{q_2} \left[-\left[(29\delta + 10)(e_2 - e_3) \right] + q_2 R_2(w) + v_2 \dot{x}_2 (\psi_2 z_2 - n_2 y_2) + v_2 x_2 (\psi_2 \dot{z}_2 - n_2 \dot{y}_2) \right], \\ \kappa_3 = -\frac{1}{q_3} \left[-2\left(\frac{\delta + 8}{3}\right) e_3 + q_3 R_3(w) + v_3 \dot{x}_3 (\psi_3 z_3 - n_3 y_3) + v_3 x_3 (\psi_3 \dot{z}_3 - n_3 \dot{y}_3) \right]. \end{cases}$$

Then:

$$\begin{aligned} \kappa_1 &= -\frac{1}{q_1} \left[-(25\delta + 10)(e_1 - e_2) + q_1 R_1(w) + v_1 (25\delta + 10)(x_2 - x_1)(\psi_1 z_1 - n_1 y_1) \right. \\ &\quad \left. + v_1 x_1 (\psi_1 (25\delta + 10)(z_2 - z_1) - n_1 (25\delta + 10)(y_2 - y_1)) \right], \\ \kappa_2 &= -\frac{1}{q_2} \left[v_2 (29\delta + 10)(e_2 - e_3) + q_2 R_2(w) + v_2 \left((-35\delta + 28)x_1 - x_1 x_3 + (29\delta - 1)x_2 \right) (\psi_2 - n_2 y_2) \right. \\ &\quad \left. + v_2 x_2 \left(\psi_1 (-35\delta + 28)z_1 - z_1 z_3 + (29\delta - 1)z_2 - n_2 \left((-35\delta + 28)y_1 - y_1 y_3 + (29\delta - 1)y_2 \right) \right) \right], \\ \kappa_3 &= -\frac{1}{q_3} \left[-2\left(\frac{\delta + 8}{3}\right) e_3 + q_3 R_3(w) + v_1 \left(x_1 x_2 - \left(\frac{\delta + 8}{3}\right) x_3 \right) (\psi_3 z_3 - n_3 y_3) \right. \\ &\quad \left. + v_3 x_3 \left(\psi_3 \left(z_1 z_2 - \left(\frac{\delta + 8}{3}\right) z_3 \right) - n_3 \left(y_1 y_2 - \left(\frac{\delta + 8}{3}\right) y_3 \right) \right) \right]. \end{aligned}$$

then:

$$\left\{ \begin{array}{l} \dot{w}_1 = R_1(w) - \frac{1}{q_1} \left[-(25\delta + 10)(e_1 - e_2) + q_1 R_1(w) + v_1 \left((25\delta + 10)(x_2 - x_1)(\psi_1 z_1 - n_1 y_1) \right. \right. \\ \left. \left. + x_1 \left(\psi_1 (25\delta + 10)(z_2 - z_1) - n_1 (25\delta + 10)(y_2 - y_1) \right) \right) \right], \\ \dot{w}_2 = R_2(w) - \frac{1}{q_2} + v_2 \left[(29\delta + 10)(e_2 - e_3) + q_2 R_2(w) + v_2 (\psi_2 - n_2 y_2) \left((-35\delta + 28)x_1 - x_1 x_3 \right. \right. \\ \left. \left. + (29\delta - 1)x_2 \right) + v_2 x_2 \left(\psi_1 (-35\delta + 28)z_1 - z_1 z_3 + (29\delta - 1)z_2 \right. \right. \\ \left. \left. - n_2 \left((-35\delta + 28)y_1 - y_1 y_3 + (29\delta - 1)y_2 \right) \right) \right], \\ \dot{w}_3 = R_3(w) - \frac{1}{q_3} \left[-2 \left(\frac{\delta+8}{3} \right) e_3 + q_3 R_3(w) + v_1 \left(x_1 x_2 - \left(\frac{\delta+8}{3} \right) x_3 \right) (\psi_3 z_3 - n_3 y_3) \right. \\ \left. + v_3 x_3 \left(\psi_3 \left(z_1 z_2 - \left(\frac{\delta+8}{3} \right) z_3 \right) - n_3 \left(y_1 y_2 - \left(\frac{\delta+8}{3} \right) y_3 \right) \right) \right]. \end{array} \right.$$

Substituting $\dot{x}, \dot{y}, \dot{z}$ and $\dot{w}$ in (5.21), we get:

$$\left\{ \begin{array}{l} \dot{e}_1 = -(25\delta + 10)(e_1 - e_2), \\ \dot{e}_2 = -(29\delta + 1)(e_2 - e_3), \\ \dot{e}_3 = -2 \left(\frac{\delta + 8}{3} \right) e_3. \end{array} \right. \quad (5.22)$$

It is easy to see that its eigenvalues are:

$$-(25\delta + 10), \quad -(29\delta + 1), \quad -2 \left(\frac{\delta + 8}{3} \right)$$

,

that are strictly negative real part . According to stability theory of linear systems with integer-order, error system (5.22) solutions error goes to zero, which means that the systems (5.17)-(5.20) are globally synchronized.

About the simulation, assume that the initial value of the three drive systems are:

$$x_1(0) = 0, \quad x_2(0) = 1 \text{ and } x_3(0) = -5,$$

$$y_1(0) = 2, \quad y_2(0) = 3 \text{ and } y_3(0) = -3,$$

$$z_1(0) = 3, \quad z_2(0) = 3 \text{ and } z_3(0) = -3,$$

the initial conditions of the response system are:

$$w_1(0) = 5, \quad w_2(0) = -5 \text{ and } w_3(0) = 5.$$

And

$$v_i = n_i = \psi_i = q_i = 2, i = 1, 2, 3,$$

Therefore, the error system initial conditions are:

$$e_1(0) = 10, e_2(0) = -10 \text{ and } e_3(0) = 10,$$

The attractors of the unified chaotic system, for different values p and δ appear in Figures 5.1, 5.2 and 5.3,

Figure 5.4 shows the evolution of synchronization errors between the three drive systems (5.17)-(5.19) and the response system (5.20), where $\delta = 1$.

Of course, each solution of the error system (5.22) goes to zero, that shows that the CCS of the systems is realized. fractional calculus and the fractional Lyapunov direct method theorem,

Chaos combination anti-synchronization (CCA) of chaotic systems affected by random noise

In view of some important concepts about the direct method of Lyapunov in fractional-order , and fractional calculus , the CCAS of chaotic systems with non integer-order problem is studied in this chapter. These systems are perturbed by some random noise. For achieving the synchronization error system's asymptotic stability , a new simple adaptive control scheme is introduced. The CCAS technique is applied for a class of some chaotic systems with non integer-order . A numerical example is provided. To confirm effectiveness of the general technique, where the synchronizations of two chaotic financial systems and one chaotic dynamo system can be achieved. The results from simulations show that the suggested techniques are reliable and efficient.

6.1 General scheme of an adaptive CCAS

Consider two drive systems as an example of chaotic systems with non integer-order , which are:

$$D^p x = f(x) + \xi_1(x)\eta + R_1(t), \quad (6.1)$$

and

$$D^p y = g(y) + \xi_2(y)\theta + R_2(t), \quad (6.2)$$

where:

* D^p is the differential operator of Caputo..

* $p \in (0, 1)$ represents the fractional-order.

* $(x_1, x_2, \dots, x_n)^T$ represent the states of the first drive, $y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n$ represent the states of the second drive systems.

* The two functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous and without uncertain parameters

* $\xi_1, \xi_2 \in \mathbb{R}^{n \times n}$ are matrix functions.

* $\eta = (\eta_1, \eta_2, \dots, \eta_m)^T \in \mathbb{R}^m$ are the uncertain parameters of the systems (6.1), $\theta = (\theta_1, \theta_2, \dots, \theta_s)^T \in \mathbb{R}^s$ of the system(6.2).

* $R_1(t)$ are the bounded noise of system (6.1) and $R_2(t)$ are the bounded noise of system(6.2), respectively.

The response system is also described by:

$$D^p z = h(z) + \xi_3(z)\vartheta + R_3(t) + u, \quad (6.3)$$

where:

* $z = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}^n$ represents the state of the system (6.3).

* $h \in \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the vector function without uncertain parameter.

* $\xi_3 \in \mathbb{R}^{n \times n}$ is matrix function.

* $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_r)^T \in \mathbb{R}^r$ is the uncertain parameters of the system (6.3) .

* $R_3(t)$ is the bounded noise of the system (6.3).

Here, we announce the definition of CCAS between the systems (6.1)-(6.2) and (6.3) as follow.

Definition 6.1.1. *The response system (6.3) and drive systems (6.1-6.2) are considered combination anti-synchronized if, with an appropriately designed adaptive controller u , we have:*

$$\lim_{t \rightarrow +\infty} \|(x + y + z)(t)\| = 0.$$

consider the state error:

$$e = x + y + z, \quad (6.4)$$

From Eqs. (6.1), (6.2) and (6.3), the error dynamical system (6.4) can be defined as

$$D^p e = f(x) + g(y) + h(z) + \xi_1(x)\tilde{\eta} + \xi_2(y)\tilde{\theta} + \xi_3(z)\tilde{\vartheta} + R(t) + u, \quad (6.5)$$

where

$$R(t) = R_1(t) + R_2(t) + R_3(t),$$

which satisfies the following condition:

$$\|R(t)\| \leq \sigma \text{ for } t \geq 0, \text{ and } \sigma > 0.$$

The issue to achieve the CCAS of chaotic systems, is now become a problem of choosing a control law u , where the error of synchronization (6.5) tends to zero.

In the following section, A simple adaptive control scheme is suggested to ensure this goal.

Theorem 6.1.1. Assume that:

1. The controller u is:

$$u = -\varphi(x, y, z) - \left(\xi_1(x)\hat{\eta} + \xi_2(y)\hat{\theta} + \xi_3(z)\hat{\vartheta} \right) - \left(\Theta + \frac{\delta}{\|e\|} \right) e, \quad (6.6)$$

where $e \neq 0$, $\varphi(x, y, z) = (f(x) + g(y) + h(z))$ and Θ represents positive matrix.

2. The update laws are:

$$D^p \hat{\eta} = [\xi_1(x)]^T e, \quad (6.7)$$

$$D^p \hat{\theta} = [\xi_2(x)]^T e, \quad (6.8)$$

and

$$D^p \vartheta = [\xi_3(x)]^T e, \quad (6.9)$$

with $\hat{\eta}$, $\hat{\theta}$ and $\hat{\vartheta}$ represent the estimations of η , θ and ϑ , respectively. The CCAS between the drive systems (6.1), (6.2) and the response system (6.3) can be attained.

Proof. According to the control (6.6), the error dynamical system is obtained by

$$D^p e = \xi_1(x)e_\eta + \xi_2(y)e_\theta + \xi_3(z)e_\vartheta + R(t) - \left(\Theta + \frac{\delta}{\|e\|} \right) e, \quad (6.10)$$

where

$$e_\eta = \eta - \hat{\eta},$$

$$e_\theta = \theta - \hat{\theta},$$

and

$$e_\vartheta = \vartheta - \hat{\vartheta}.$$

Define the Lyapunov functional as

$$V = \frac{1}{2} (e^T e + e_\eta^T e_\eta + e_\theta^T e_\theta + e_\vartheta^T e_\vartheta).$$

By using Lemma 3.2.1, we have:

$$\begin{aligned} D^p V &\leq e^T D^p e + e_\eta^T D^p e_\eta + e_\theta^T D^p e_\theta + e_\vartheta^T D^p e_\vartheta \\ &= e^T \left(\xi_1(x) e_\eta + \xi_2(y) e_\eta + \xi_3(z) e_\vartheta + R(t) - \Theta e - \frac{\delta}{\|e\|} e \right) + \\ &\quad - e_\eta^T [\xi_1(x)]^T e - e_\theta^T [\xi_2(x)]^T e - e_\vartheta^T [\xi_3(x)]^T e \\ &\leq \delta \|e\| - e^T \Theta e - \delta \|e\| \\ &= -e^T K e \\ &< 0, \text{ for all } e \neq 0. \end{aligned} \tag{6.11}$$

According to Theorem 3.2.3, it can be remarked that the system (6.10) asymptotically converges to zero, that indicate that the combination between the systems (6.1),(6.2)and (6.3) achieve CCAS. $\square$

Remark 6.1.1. *Indeed, the choice of the previous control (6.6) poses a problem of the non-continuity of the control u at point 0. It will then have an obstacle for our algorithm, that we will use for the solutions of the considered systems. To avoid this problem, we will replace the control (6.6)(without changing the previous theorem's proof) by*

$$u = -\varphi(x, y, z) - \left(\xi_1(x) \hat{\eta} + \xi_2(y) \hat{\eta} + \xi_3(z) \hat{\vartheta} \right) - \left(\Theta + \frac{\delta}{\|e\| + \epsilon(t)} \right) e, \tag{6.12}$$

where ϵ is a small enough positive function, wich tends to 0, when t is large enough.

6.2 Illustrative example and simulation results

6.2.1 Illustrative example

The financial system with fractional-order and the modified coupled dynamos system are used as examples in this section to confirm the efficacy and efficiency of the suggested strategies..

Assumed is the fractional-order financial systems [50] represent the two drive systems and the modified system of coupled dynamos [51] represents the response system.

The dynamic equations of the drive systems respectively,are

$$\begin{cases} D^p x_1 = x_3 - ax_1 + x_1 x_2 + r_1(t), \\ D^p x_2 = 1 - bx_2 - x_1^2 + r_2(t), \\ D^p x_3 = -x_1 - cx_3 + r_3(t), \end{cases} \quad (6.13)$$

and

$$\begin{cases} D^p y_1 = y_3 - ay_1 + y_1 y_2 + r_4(t), \\ D^p y_2 = 1 - by_2 - y_1^2 + r_5(t), \\ D^p y_3 = -y_1 - cy_3 + r_6(t). \end{cases} \quad (6.14)$$

The modified coupled dynamo system [51] is given by

$$\begin{cases} D^p z_1 = -\alpha z_1 + z_2 (z_3 + \beta) + r_7(t) + u_1, \\ D^p z_2 = -\alpha z_2 + z_1 (z_3 - \beta) + r_8(t) + u_2, \\ D^p z_3 = -z_1 z_2 + z_3 + r_9(t) + u_3, \end{cases} \quad (6.15)$$

where $((r_1, r_2, r_3)(t))^T$, $((r_4, r_5, r_6)(t))^T$ and $((r_7, r_8, r_9)(t))^T$ are the bounded noise of systems (6.13), (6.14) and (6.15), respectively, i.e., $\exists \sigma_i > 0, i = 1, 2, 3$, such that:

$$\begin{cases} \|r_1(t) + r_4(t) + r_7(t)\| \leq \sigma_1, \text{ for all } t \\ \|r_2(t) + r_5(t) + r_8(t)\| \leq \sigma_2, \text{ for all } t \\ \|r_3(t) + r_6(t) + r_9(t)\| \leq \sigma_3, \text{ for all } t. \end{cases}$$

Now, we define the combination anti-synchronization error as

$$\begin{cases} e_1 = x_1 + y_1 + z_1, \\ e_2 = x_2 + y_2 + z_2, \\ e_3 = x_3 + y_3 + z_3, \end{cases}$$

Define the error estimate parameters as:

$$e_a = a - \hat{a},$$

$$e_b = b - \hat{b},$$

$$e_c = c - \hat{c},$$

$$e_\alpha = \alpha - \hat{\alpha}$$

and

$$e_\beta = \beta - \hat{\beta},$$

such that $\hat{a}, \hat{b}, \hat{c}, \hat{\alpha}, \hat{\beta}$ represent the parameters for the estimate of a, b, c, α, β , respectively.

By using Theorem 6.1.1, one gets:

$$\begin{cases} u_1 = -\theta(x, y) + \hat{a}(x_1 + y_1) + \hat{\alpha}z_1 - z_2(z_3 + \hat{\beta}) - \left(k_1 + \frac{\delta_1}{|e_1|}\right)e_1, \\ u_2 = \hat{b}(x_2 + y_2) + x_1^2 + y_2^2 + \hat{\alpha}z_2 - z_1(z_3 - \hat{\beta}) - \left(k_2 + \frac{\delta_2}{|e_2|}\right)e_2, \\ u_3 = (x_1 + y_1) + \hat{c}(x_3 + y_3) + z_1z_2 - z_3 - \left(k_3 + \frac{\delta_3}{|e_3|}\right)e_3, \end{cases} \quad (6.16)$$

where $\theta(x, y) = x_3 + y_3 + x_1x_2 + y_1y_2$, and

$$\begin{cases} D^p \hat{a} = -e_1(x_1 + y_1), \\ D^p \hat{b} = -e_2(x_2 + y_2) \\ D^p \hat{c} = -e_3(x_3 + y_3) \\ D^p \hat{\alpha} = -e_1z_1 - e_2z_2, \\ D^p \hat{\beta} = e_1z_2 - e_2z_1. \end{cases} \quad (6.17)$$

$$\begin{cases} D^p e_1 = -e_a(x_1 + y_1) - e_\alpha z_1 + e_\beta z_2 + R_1(t) - \left(k_1 + \frac{\delta_1}{|e_1|}\right)e_1, \\ D^p e_2 = -e_b(x_2 + y_2) - e_\alpha z_2 - e_\beta z_1 + R_2(t) - \left(k_2 + \frac{\delta_2}{|e_2|}\right)e_2, \\ D^p e_3 = -e_c(x_3 + y_3) + R_3(t) - \left(k_3 + \frac{\delta_3}{|e_3|}\right)e_3, \end{cases} \quad (6.18)$$

$$\begin{cases} R_1(t) = r_1(t) + r_4(t) + r_7(t), \\ R_2(t) = r_2(t) + r_5(t) + r_8(t), \\ R_3(t) = r_3(t) + r_6(t) + r_9(t). \end{cases} \quad (6.19)$$

Define the Lyapunov function as;

$$V = 0.5 \sum_{i=1}^3 e_i^2 + e_a^2 + e_b^2 + e_c^2 + e_\alpha^2 + e_\beta^2. \quad (6.20)$$

According to Lemma 3.2.1, one has

$$\begin{aligned}
D^p V &\leq \sum_{i=1}^3 e_i D^p e_i + e_a D^p e_a + e_b D^p e_b + e_c D^p e_c \\
&\quad + e_\alpha D^p e_\alpha + e_\beta D^p e_\beta \\
&= e_1(-e_a(x_1 + y_1) - e_\alpha z_1 + e_\beta z_2 + R_1(t) - (k_1 + \frac{\delta_1}{|e_1|})e_1 \\
&\quad + e_2(-e_b(x_2 + y_2) - e_\alpha z_2 - e_\beta z_1 + R_2(t) - (k_2 + \frac{\delta_2}{|e_2|})e_2) \\
&\quad + e_3(-e_c(x_3 + y_3) + R_3(t) - (k_3 + \frac{\delta_3}{|e_3|})e_3) \\
&\quad - (e_a D^p \hat{a} + e_b D^p \hat{b} + e_c D^p \hat{c} + e_\alpha D^p \hat{\alpha} + e_\beta D^p \hat{\beta}) \\
&\leq -\sum_{i=1}^3 k_i e_i^2 + \sum_{i=1}^3 \delta_i |e_i| + e_a(-e_1(x_1 + y_1) - D^p \hat{a}) \\
&\quad + e_b(-e_2(x_2 + y_2) - D^p \hat{b}) + e_c(-e_3(x_3 + y_3) - D^p \hat{c}) \\
&\quad + e_\alpha(-e_1 z_1 - e_2 z_2 - D^p \hat{\alpha}) + e_\beta(e_1 z_2 - e_2 z_1 - D^p \hat{\beta}) \\
&\quad - \sum_{i=1}^3 \delta_i \frac{e_i^2}{|e_i|}, e_i \neq 0, i = 1, 2, 3 \\
&\leq -\sum_{i=1}^3 k_i e_i^2 < 0.
\end{aligned}$$

By Theorem 3.2.3, we can conclude that the error systems (6.18) tends to 0, which means that CCAS between systems (6.13), (6.14) and (6.15) can be satisfied, under the controller (6.16).

Remark 6.2.1. As we mentioned in Remark 6.1.1, the choice of the previous controls (6.16) poses a problem at the zero point. To avoid this problem, the controllers (6.16) can be replaced by

$$\left\{ \begin{array}{l} u_1 = -(x_3 + y_3) + \hat{a}(x_1 + y_1) - x_1 x_2 - y_1 y_2 + \hat{\alpha} z_1 - z_2(z_3 + \hat{\beta}) \\ \quad - (k_1 + \frac{\delta_1}{|e_1| + 10^{-(t+10)}})e_1, \\ u_2 = \hat{b}(x_2 + y_2) + x_1^2 + y_2^2 + \hat{\alpha} z_2 - z_1(z_3 - \hat{\beta}) \\ \quad - (k_2 + \frac{\delta_2}{|e_2| + 10^{-(t+10)}})e_2, \\ u_3 = (x_1 + y_1) + \hat{c}(x_3 + y_3) + z_1 z_2 - z_3 \\ \quad - (k_3 + \frac{\delta_3}{|e_3| + 10^{-(t+10)}})e_3. \end{array} \right. \quad (6.21)$$

Therefore, the error dynamics becomes

$$\begin{cases} D^p e_1 = -e_a(x_1 + y_1) - e_\alpha z_1 + e_\beta z_2 + R_1(t) \\ \quad - (k_1 + \frac{\delta_1}{|e_1| + 10^{-(t+10)}})e_1, \\ D^p e_2 = -e_b(x_2 + y_2) - e_\alpha z_2 - e_\beta z_1 + R_2(t) \\ \quad - (k_2 + \frac{\delta_2}{|e_2| + 10^{-(t+10)}})e_2, \\ D^p e_3 = -e_c(x_3 + y_3) + R_3(t) \\ \quad - (k_3 + \frac{\delta_3}{|e_3| + 10^{-(t+10)}})e_3, \end{cases} \quad (6.22)$$

6.2.2 Numerical simulations

Here, the method of Adams–Bashforth–Moulton [37] is employed to solve these systems. We suppose that the starting values the starting values of the drive systems are employed, respectively, as:

$$x_1(0) = -3, x_2(0) = -3 \text{ and } x_3(0) = 3,$$

$$y_1(0) = -3, y_2(0) = -3 \text{ and } y_3(0) = 3.$$

the initial conditions of the response system are:

$$z_1(0) = -3, z_2(0) = -3 \text{ and } z_3(0) = 3.$$

Hence, the initial conditions of the error system are :

$$e_1(0) = -9, e_2(0) = -9, e_3(0) = 9,$$

the bounded noise are selected as

$$r_1(t) = 0.01 \sin(\pi t),$$

$$r_2(t) = 0.01 \sin(\pi t),$$

$$r_3(t) = 0.01 \sin(\pi t),$$

$$r_4(t) = 0.02 \cos(2\pi t),$$

$$r_5(t) = 0.02 \cos(2\pi t),$$

$$r_6(t) = 0.02 \cos(2\pi t),$$

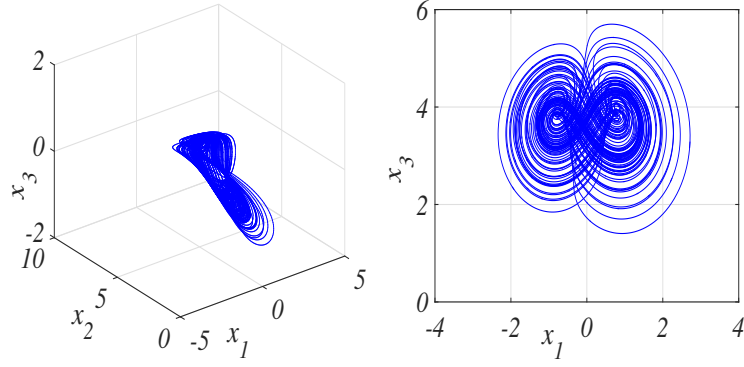


Figure 6.1: Typical chaotic system dynamical behaviors (6.13).

$$r_7(t) = 0.03 \sin(3\pi t),$$

$$r_8(t) = 0.03 \sin(3\pi t),$$

$$r_9(t) = 0.03 \sin(3\pi t).$$

To ensure the condition of stability, the control inputs are chosen as:

$$(k_1, k_2, k_3) = (0.5, 0.6, 0.7),$$

the unidentified parameters of systems (6.13) and (6.14) are taken as:

$$a = 3, b = 0.1, c = 1,$$

the unknown parameters of system (6.15) are chosen as:

$$\alpha = 2, \beta = 1,$$

Also:

$$\hat{a}(0) = 2, \hat{b}(0) = 0.2, \hat{c}(0) = 0.5,$$

$$\hat{\alpha}(0) = 1 \text{ and } \hat{\beta}(0) = 0.5,$$

and finally $p = 0.98$.

The projections of fractional-order financial chaotic system (6.13) without random noise $(r_1(t), r_2(t), r_3(t))$, are shown in Figure 6.1.

The modified coupled dynamos system (5.20) exhibits a chaotic behavior without random noise $r_7(t), r_8(t), r_9(t)$ and controllers (u_1, u_2, u_3) , which is represent in Figure 6.2.

Figure 6.3 give us the temporal evolution of synchronization error without the controller (6.16) and adaptive law (6.17).

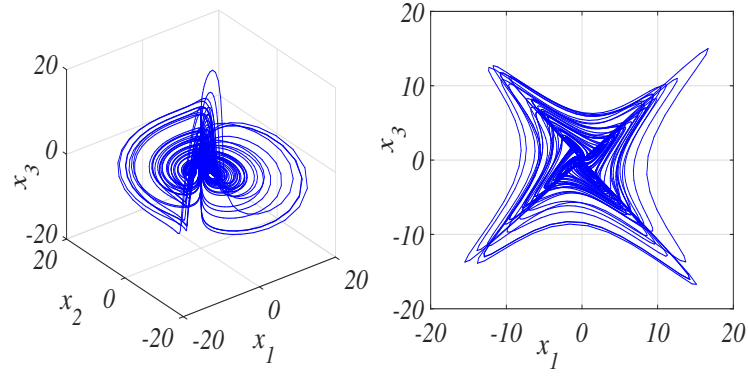


Figure 6.2: Typical chaotic system dynamical behaviors (6.15).

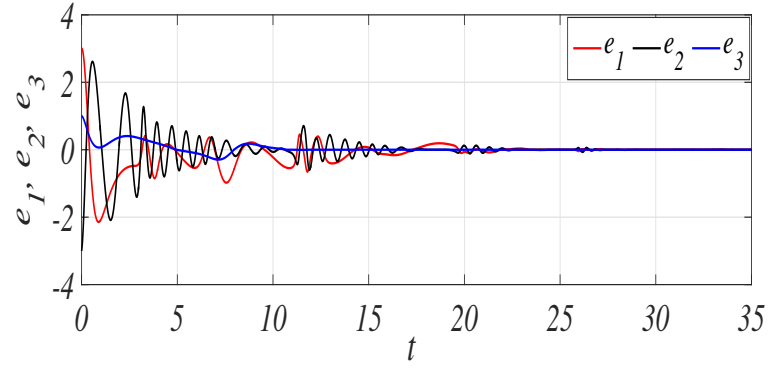


Figure 6.3: The time responses of error system(6.22).

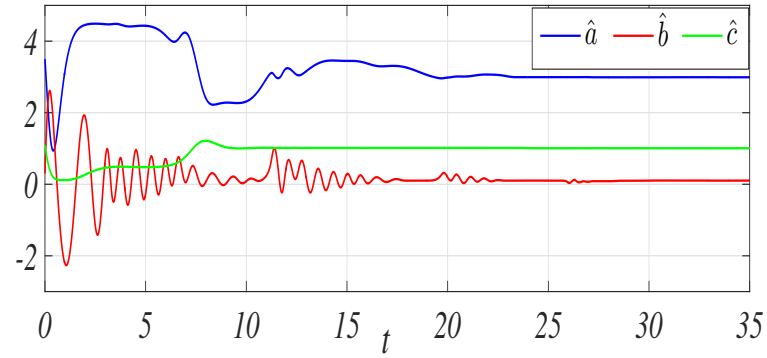


Figure 6.4: Time response of the update parameters of system (6.13).

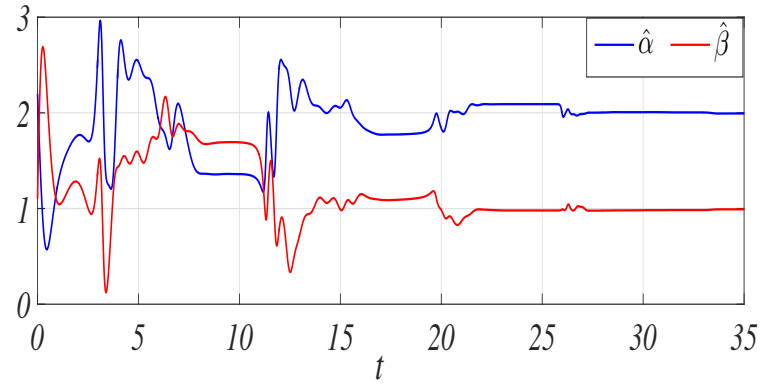


Figure 6.5: Time response of the update parameters of system (6.15).

Figures 6.4 and 6.5 represent the temporal evolution of the estimate parameter systems. Obviously, each of the estimated parameters and the errors of synchronization tend to the original values and 0, respectively, that means that the CCAS of the proposed systems is achieved.

Conclusion

In this thesis work, we have presented different strategies about the combination synchronization of certain chaotic systems with fractional-order.

In the first chapter, we have presented certain concepts notions of chaotic dynamical systems, which we believe are useful for a good understanding of our research topic presented in this thesis.

In the second chapter, we have shown some basic notions of fractional calculus. We have also justified our choice of the Caputo derivation in the work presented in this thesis. We have then described in the third chapter the equations with fractional-order and the stabilization chaotic systems with fractional by the approach of a new extension of the direct method of Lyapounov of fractional-order. The use of this new approach has allowed us to show under certain adequate hypotheses the demonstration of the asymptotic stabilization of such systems.

The fourth and fifth chapters are mainly devoted to the study of two major subjects. In the first topic, the issue of compound combination synchronization between three integer-order united chaotic systems and one fractional-order united chaotic system has been considered. While in the second topic, a novel method to study the issue of chaos combination anti-synchronization of three fractional order chaotic systems perturbed by random noise has been developed and a suitable Lyapunov function has been adopted to perform the stability in addition to the convergence of the synchronization error.

Finally, the validity and capability of the suggested synchronization techniques have been tested and verified by numerical simulations.

Although the work presented in this thesis is limited to simple access, our fundamental results could therefore give rise to further studies in the following directions:

* Penta compound combination synchronization of some chaotic systems with non integer-order with analysis and application.

*Generalized the compound combination synchronization to n - non integer-order chaotic systems.

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