الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
République Algérienne Démocratique et Populaire
وزارة التعليـم العالـي والبحـث العلمـي
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Centre Universitaire de MILA

N° Réf :...............

Centre Universitaire
Abd Elhafid Boussouf Mila

Institut des Sciences et Technologie
Département de Mathématiques et Informatique

Mémoire préparé en vue de l'obtention du diplôme de Master

En : Mathématiques

Spécialité : Mathématiques Appliquées

Étude Numérique des Équations Intégrales de Volterra de Deux Dimensions

Préparé par:

Birem Fouzia

Soutenue devant le jury:

BOULMERKA Aissa (MCA)	C U Abd Elhafid Boussouf	Président
BELLOUR Azzeddine (Prof.)	ENS de constantine	Examinateur
LAIB Hafida (MCB)	C U Abd Elhafid Boussouf	Rapporteur

Année Universitaire : 2020/2021

People´ s Democratic Republic of Algeria

Ministry of Higher Education and scientific Research

University Center Abd Elhafid Boussouf MILA

Institute of science and technology Department of mathematics and Computing

Dissertation prepared for graduation from Master

In Applied Mathematics

NUMERICAL STUDY OF TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

By : Fouzia Birem

University year : 2020 – 2021

ACKNOWLEDGMENTS

I would like to thank Allah first and foremost. My deepest gratitude goes to my supervisor **Dr. LAIB Hafida** for the continuous support of my research, for her patience, motivation, enthusiasm, and her wise advice.
I would express my gratitude to the members of the jury namely **Prof. BELLOUR Azzeddine** and **Dr. BOULMERKA Aissa** for their intellectual insights, encouragement and unconditioned help.
Special thanks to the head of the Department of Mathematics and Computing.
I also want to show my cordial respect and appreciation to all of my teachers and classmates.

DEDICATION

With the feelings of love and gratitude I dedicate this modest work :

-To my beloved mother and to the soul of my father may Allah bless him.
-To my husband for his unconditioned support and encouragement and to my kids for their inspiration.
-To my family, thank you for being always on there for me.
-To everyone who encouraged me.

Fouzia

ملخص

في هذه المذكرة، تم تطبيق طريقة التجميع لتايلور للحصول على الحل التقريبي لمعادلات فولتيرا التكاملية ذو البعد الثاني مع تطوير خوارزمية باستخدام كثيرات حدود تايلور لإيجاد الحل العددي التقريبي لهذا النوع من المعادلات، كما تم إدراج أمثلة عددية لتأكيد النتائج النظرية و لإثبات تقارب الخوارزمية المقدمة.

الكلمات المفتاحية: معادلات فولتيرا التكاملية ذو البعد الثاني، طريقة التجميع، كثيرات حدود تايلور.

ABSTRACT

In this dissertation , Taylor collocation method is applied to obtain the approximate solution for two-dimensional linear Volterra integral equations. An algorithm based on the use of Taylor polynomials is developed for the numerical solution of this kind of equations. We also provide a rigorous error analysis which justifies that the errors of the approximate solution of the exact solution. Numerical examples are included to prove the validity and the efficacy of the presented convergent algorithm.

Key Words: Two-dimensional Volterra integral equations; Collocation method; Taylor polynomials.

RÉSUMÉ

Dans ce mémoire, la méthode de " Taylor collocation " est appliquée pour obtenir la solution approchée des équations intégrales de deux dimensions de Volterra. Un algorithme s'est basé sur l'utilisation des polynômes de Taylor sont développés pour la solution numérique de ce type d'équations. Des exemples numériques sont présentés pour confirmer les estimations théoriques et illustrer la convergence de la méthode.

Mots-clés: Équations intégrales de Volterra de deux dimensions; Méthode de collocation; Polynômes de Taylor.

CONTENTS

INTRODUCTION

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. The first integral equation mentioned in the mathematical literature is due to Abel. He found this equation in 1823, starting from a problem in mechanics. He gave a very elegant solution that was published in 1826.

Starting in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators. In 1900, Ivar Fredholm made his famous contribution that led to a fascinating period in the development of mathematical analysis. Poincaré, Fréchet, Hilbert, Schmidt, Hardy and Riesz were involved in this new area of research.

Two-dimensional integral equations have significant applications in various fields of applied science and engineering such as plasma physics, the image deblurring problem and its regularization, axisymmetric contact problems for bodies with complex rheology, diffraction theory and the electrochemical behavior of an inlaid microband electrode for the case of equal diffusion coefficients. These types of integral equations also occur as reformulations from some mixed boundary value problems arising in various branches of applied sciences; for example, solid and fluid mechanics, electrostatics, heat transfer, diffraction and scattering of waves, etc.

The main purpose of this work is to find an approximate solution for linear two-dimensional Volterra integral equations (2D-VIEs) based on the Taylor collocation method (TCM).

The collocation method is based on the idea of approaching the exact solution of a given integral equation using a suitable function belonging to a chosen finite-dimensional space. The approximate solution must satisfy the integral equation on a certain subset of the interval (called the set of collocation points). We consider as space of approximation the space of the spline polynomials : $S^{(-1)}_{p-1,p-1}(\Pi_{N,M})$ defined as in (2.1.2). The main advantages of this method are :

- This method is direct and the approximate solution is given by using explicit formulas;
- This method has a convergence order;
- There is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

Our dissertation is organized as follows :

- The first chapter : We provide the fundamental notions, definitions and some necessary theorems will be needed for the following chapter, such as the Taylor series, Leibnitz rule, and Comparison theorems...
- The second chapter : We give an algorithm for solving linear 2D-VIEs using the Taylor collocation method. We prove the convergence of the approximate solution to the exact solution.
- The third chapter : We give some numerical examples to illustrate the theoretical results obtained in the previous chapter. In each example, we calculate the error for different values of N, M and p between u and the Taylor collocation solution v. We compare our results with another method.

CHAPTER 1

PRELIMINARY AND AUXILIARY RESULTS

In this chapter, we define some necessary notions and theories, such as the Taylor series, integral equations and comparison theorems which will demonstrate the convergence of the approximate solutions.

1.1 Taylor series

Definition 1.1.1 *For a function of two variables $f(x, y)$ whose partials all exist to the n^{th} partials at the point (a, b), the n^{th}-degree Taylor polynomial of f for (x, y) near the point (a, b) is :*

$$P_n(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{1}{i!j!} \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} (x-a)^i (y-b)^j \ .$$

Example 1.1.1 *For a function of two variables $f(x, y) = xe^y + 1$ whose first, second and third partials exist at the point $(1, 0)$. The third-degree Taylor polynomial of f around the point $(1, 0)$ is :*

$$\begin{aligned}
p_3(x,y) =& f(1,0) + \frac{\partial f(1,0)}{\partial y} y + \frac{\partial f(1,0)}{\partial x}(x-1) \\
&+ \frac{1}{2!}\frac{\partial^2 f(1,0)}{\partial y^2} y^2 + \frac{\partial^2 f(1,0)}{\partial x \partial y}(x-1)y + \frac{1}{2!}\frac{\partial^2 f(1,0)}{\partial x^2}(x-1)^2 \\
&+ \frac{1}{3!}\frac{\partial^3 f(1,0)}{\partial y^3} y^3 + \frac{1}{2!}\frac{\partial^3 f(1,0)}{\partial x \partial y^2}(x-1)y^2 + \frac{1}{2!}\frac{\partial^3 f(1,0)}{\partial x^2 \partial y}(x-1)^2 y + \frac{1}{3!}\frac{\partial^3 f(1,0)}{\partial x^3}(x-1)^3 \\
=& 1 + x + \frac{1}{2}y + \frac{1}{2}y^2 + \frac{1}{2}x^2 y + \frac{1}{6}y^3.
\end{aligned}$$

Theorem 1.1.1 (*Taylor's Theorem for functions of two independent variables[5]*) *Let f be p times continuously differentiable on $D = [a, b] \times [c, d]$ and let $(x_0, y_0) \in D$. Then for all $(x, y) \in D$, we have*

$$f(x,y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}(x-x_0)^i (y-y_0)^j + \sum_{i+j=p} \frac{1}{i!j!} \frac{\partial^{i+j} f(x_1,y_1)}{\partial x^i \partial y^j}(x-x_0)^i (y-y_0)^j,$$

where

$$\begin{cases} x_1 = \theta x + (1-\theta)x_0 \in [a,b], \\ y_1 = \theta y + (1-\theta)y_0 \in [c,d], \end{cases} \qquad \theta \in (0,1).$$

1.2 Classification of integral equations

An integral equation is the equation in which the unknown function $u(x,y)$ appears inside an integral. The most standard type of two-dimensional integral equation in $u(x,y)$ is of the form :

$$u(x,y) = f(x,y) + \lambda \int_{g(x)}^{h(x)} \int_{k(y)}^{l(y)} K(x,y,t,s,u(t,s))dsdt, \tag{1.2.1}$$

and if equation (1.2.1) is linear then it becomes

$$u(x,y) = f(x,y) + \lambda \int_{g(x)}^{h(x)} \int_{k(y)}^{l(y)} K(x,y,t,s)u(t,s)dsdt,$$

where $g(x), h(x), k(y)$ and $l(y)$ are the limits of integration, may be variables, constants, or mixed, λ is a constant parameter, and $K(x,y,t,s)$ is a known function, called the kernel or the nucleus of the integral equation. The unknown function $u(x,y)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x,y)$ appears inside and outside the integral sign. The functions $f(x,y)$ and $K(x,y,t,s)$ are given in advance. Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations.

Fredholm integral equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x, y)$ may appear only inside integral equation in the form :

$$f(x, y) = \lambda \int_a^b \int_c^d K(x, y, t, s)u(t, s)dsdt.$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x, y)$ appears inside and outside the integral sign. The second kind is represented by the form :

$$u(x, y) = f(x, y) + \lambda \int_a^b \int_c^d K(x, y, t, s)u(t, s)dsdt.$$

Volterra integral equations

In Volterra integral equations, at least one of the limits of each integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x, y)$ appears only inside integral sign in the form :

$$f(x, y) = \int_a^x \int_c^y K(x, y, t, s)u(t, s)dsdt.$$

However, Volterra integral equations of the second kind, the unknown function $u(x, y)$ appears inside and outside the integral sign. The second kind is represented by the form :

$$u(x, y) = f(x, y) + \lambda \int_a^x \int_c^y K(x, y, t, s)u(t, s)dsdt.$$

Volterra-Fredholm integral equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models.

The Volterra-Fredholm integral equations appear in the literature in two forms :

$$u(x,y) = f(x,y) + \lambda_1 \int_a^x \int_b^y K_1(x,y,t,s)u(t,s)dsdt + \lambda_2 \int_c^d \int_e^f K_2(x,y,t,s)u(t,s)dsdt,$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau \ \ ;(x,t) \in \Omega \times [0;T],$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0;T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$. It is interesting to note that the first equation contains disjoint Volterra and Fredholm integral equations, whereas the second equation contains mixed Volterra and Fredholm integral equations.

Remark 1.2.1 *Integral equations of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x,y)$ in the second kind of Volterra or Fredholm integral equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only.*

Examples 1.2.1 *To clarify these concepts we consider the following equations :*

1. Linear Fredholm integral equation of the first kind

$$2 - (x-y)(e^2+1) - 2e(x+y+1) = \int_0^1 \int_0^1 (s+t+x+y)u(t,s)dsdt.$$

2. Linear Fredholm integral equation of the second kind (inhomgeneous)

$$u(x,y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(8+y)} + \int_0^1 \int_0^1 \frac{x}{(8+y)(1+t+s)} u(t,s)dsdt,$$

where $(x,y) \in [0,1) \times [0,1)$.

3. Linear Volterra integral equation of the first kind

$$\frac{x^2y^2 + 2\sin(xy) - 2xy\cos(xy)}{2y^2} sin(y) = \int_0^x \int_0^y (\sin(xy)+1)u(t,s)dsdt,$$

where $(x,y) \in [0,1] \times [0,1]$.

4. Linear Volterra integral equation of the second kind (homogeneous)

$$u(x,y) = \int_0^x \int_0^y (e^{(x)t} - 1)u(t,s)dsdt,$$

where $(x,y) \in [0,1] \times [0,1]$.

5. Nonlinear Volterra integral equation of the second kind (inhomogeneous)

$$u(x,y) = x^2 + y^2 - xy\frac{xy(9x^4 + 10x^2y^2 + 9y^4)}{45} + \int_0^x \int_0^y u^2(t,s)dsdt,$$

where $(x,y) \in [0,1] \times [0,1]$.

6. Mixed Volterra and Fredholm integral equation of the second kind

$$u(x,y) = y^2 - \frac{13}{15}xy - \frac{2}{15}x^4y + \int_{-1}^x \int_{-1}^1 xyt^2s^2u(t,s)dsdt.$$

7. Disjoint Volterra and Fredholm integral equation of the second kind

$$u(x,y) = 2e^{x+y+4} - 1 + \int_0^x \int_0^y u(t,s)dsdt + 16\int_0^1 \int_0^1 e^{x+y+s+t}[u(t,s)]^3 dsdt.$$

1.3 Leibniz rule for differentiation of integrals

Let $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous in a domain of the $x - t$ plane that includes the rectangle $a \leq x \leq b$, $t_0 \leq t \leq t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)dt, \tag{1.3.1}$$

then differentiation of the integral in (1.3.1) exists and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}dt. \tag{1.3.2}$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (1.3.2) reduces to :

$$F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,t)}{\partial x} dt, \tag{1.3.3}$$

which means that differentiation and integration can be interchanged such as

$$\frac{d}{dx} \int_a^b e^{xt} dt = \int_a^b t e^{xt} dt.$$

We illustrate the Leibnitz rule by the following examples :

Example 1.3.1 *Consider the integral equation*

$$F(x) = \int_x^{x^2} \ln(1 + t^2) dt,$$

by using the rule of Leibnitz, we find

$$F'(x) = 2x ln(1 + x^4) - ln(1 + x^2).$$

Example 1.3.2 *Consider the integral equation*

$$F(x, y) = \int_0^x \int_0^y xy e^{t-s} F(t, s) ds dt, \tag{1.3.4}$$

we differentiate equation(1.3.4) *with respect to y, we obtain*

$$\frac{\partial F(x, y)}{\partial y} = \int_0^x xy e^{t-y} F(t, y) dt + \int_0^x \int_0^y x e^{t-s} F(t, s) ds dt. \tag{1.3.5}$$

Now, we differentiate equation(1.3.5) *with respect to x, we obtain*

$$\begin{aligned} \frac{\partial^2 F(x, y)}{\partial x \partial y} =& xy e^{x-y} F(x, y) + \int_0^x y e^{t-y} F(t, y) dt \\ &+ \int_0^y x e^{x-s} F(x, s) ds + \int_0^x \int_0^y e^{t-s} F(t, s) ds dt. \end{aligned}$$

1.4 Piecewise polynomial spaces

Let $\Pi_N = \{t_n : 0 = t_0 < t_1 < ... < t_N = T\}$, denote a mesh (grid) on the given interval $I = [0, T]$, where the stepsize is given by $h = \frac{T}{N}$.
Define the subintervals $\sigma_n = [t_n, t_{n+1}]$ for $n = 0, 1, ..., N - 1$.

Definition 1.4.1 *For a given mesh Π_N the piecewise polynomial space $S_\mu^{(d)}(\Pi_N)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(\Pi_N) = \{v \in C^d(I) : v|_{\sigma_n} \in \pi_\mu(0 \leq n \leq N - 1)\}.$$

Here, π_μ denotes the space of (real) polynomials of degree not exceeding μ.
It is readily verified that $S_\mu^{(d)}(\Pi_N)$ is a (real) linear vector space whose dimension is given by

$$\dim S_\mu^{(d)}(\Pi_N) = N(\mu - d) + d + 1.$$

Remark 1.4.1 *The choice of the degree of regularity d will be governed by the number of prescribed initial conditions, for Volterra integral equations (no initial condition) we choose $d = -1$.*
The particular piecewise polynomial space $S_{p-1,p-1}^{(-1)}(\Pi_{N,M})$ of bivariate polynomial spline functions of order p (degree p-1) in x and order q (degree q-1) in y is a tensor-product space based on the univariate spline spaces $S_p^{(-1)}(\Pi_N)$ and $S_q^{(-1)}(\Pi_M)$. An element of this space has jumped discontinuities at the interior grid lines $x = x_n(n = 1, ..., N - 1)$ and $y = y_m(m = 1, ..., M - 1)$.

1.5 Collocation method

A collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space such that the approximated solution satisfies the integral equation on a certain subset of the interval on which the equation has to be solved (called the set of collocation points). Here We consider as the approximating space the polynomial spline space. In order to describe the relevant collocation method for given N , let Π_N be a uniform partition of a bounded interval $I = [0, T]$ with gride points $t_n = nh, n = 0, 1, ..., N$, where $h = \frac{T}{N}$ be the stepsize. Define the subintervals $\delta_n = [t_n, t_{n+1}]\,, n = 0, ..., N-1$.

We define the real polynomial spline space of degree $p-1$ as follow :

$$S^{(-1)}_{p-1}(\Pi_N) = \{v : v_n = v|_{D_n} \in \pi_{p-1}, n = 0, ..., N-1\}.$$

The main advantages of Taylor collocation method are :
i) This method is direct and the approximate solution is given by using explicit formulas.
ii) This method has a convergence order.
iii) There is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

1.6 Comparison theorems

Lemma 1.6.1 (*Wendroff's inequality [9]*) *Let f and u be continuous and non-negative functions defined on $[a, b] \times [c, d]$, and let g be non-negative and two continuously differentiable function on $[a, b] \times [c, d] \times [a, b] \times [c, d]$. Then the inequality*

$$u(x, y) \leq f(x, y) + \int_a^x \int_c^y g(x, y, t, s)u(t, s)dsdt, \quad (x, y) \in [a, b] \times [c, d],$$

implies that

$$u(x, y) \leq f(x, y) \exp\left(\int_a^x \int_c^y R(t, s)dsdt\right), \quad (x, y) \in [a, b] \times [c, d],$$

where

$$R(x, y) = g(x, y, x, y) + \int_a^x \partial_1 g(x, y, t, y)dt$$

$$+\int_b^y \partial_2 g(x,y,x,s)ds + \int_a^x \int_c^y \partial_2\partial_1 g(x,y,t,s)dsdt.$$

Lemma 1.6.2 (*Discrete Gronwall-type inequality [3]*) *Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le p_0$ and*

$$\varepsilon_n \le p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \ge 1,$$

with $p_0 \ge 0$. Then ε_n can be bounded by

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

Lemma 1.6.3 (*Sugiyama's inequality [11]*) *Let $\varepsilon(n), a(n)$ and $b(n)$ be non-negative sequences. If $\varepsilon(n)$ satisfies*

$$\varepsilon(n) \le a(n) + \sum_{s=0}^{n-1} b(s)\varepsilon(s),$$

for all $n \in \mathbb{N}$. Then

$$\varepsilon(n) \le a(n) + \sum_{s=0}^{n-1} a(s)b(s) \prod_{\sigma=s+1}^{n-1} [1 + b(\sigma)], \quad n \in \mathbb{N}.$$

Lemma 1.6.4 (*Pachpatte's inequality [10]*) *Let $\varepsilon(n,m), a(n,m)$ and $b(n,m)$ be non-negative sequences such that $a(n,m)$ is nondecreasing in each variable n and m. If $\varepsilon(n,m)$ satisfies*

$$\varepsilon(n,m) \le a(n,m) + \sum_{s=0}^{n-1}\sum_{t=0}^{m-1} b(s,t)\varepsilon(s,t),$$

for all $n, m \in \mathbb{N}$. Then

$$\varepsilon(n,m) \le a(n,m) \prod_{s=0}^{n-1} \left[1 + \sum_{t=0}^{m-1} b(s,t)\right], \quad n, m \in \mathbb{N}.$$

CHAPTER 2

DESCRIPTION, STUDY OF CONVERGENCE AND ERROR OF THE NUMERICAL METHOD

In this chapter, we apply a direct collocation method based on the use of Taylor polynomials to approximate the solution of linear two-dimensional Volterra integral equations in the polynomial spline $S^{(-1)}_{p-1,p-1}(\Pi_{N,M})$. The approximate solution is given by using iterative formulas, and we prove the convergence of the approximate solution to the exact solution.

2.1 Description of the method

We consider a linear two-dimensional Volterra integral equation of the form :

$$u(x,y) = g(x,y) + \int_0^x \int_0^y K(x,y,t,s)u(t,s)dsdt, \quad (x,y) \in D, \tag{2.1.1}$$

where the functions g and K are given (real-valued) continuous functions defined, respectively, on $D := [0,a] \times [0,b] \subset \mathbb{R}^2$ and $S := \{(x,y,t,s) : 0 \le t \le x \le a, 0 \le s \le y \le b\}$. It follows from the classical theory of Volterra that (2.1.1) possesses a unique solution $u \in C(D)$.

Let $\Pi_N = \{x_i = ih, i = 0,1,...,N\}$ and $\Pi_M = \{y_j = jk, j = 0,1,...,M\}$ denote, respectively, uniform partitions of the intervals $[0,a]$ and $[0,b]$ with the stepsizes are given by $h = \frac{a}{N}$ and $k = \frac{b}{M}$. These partitions defined a grid for D

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(x_n, y_m), 0 \le n \le N, 0 \le m \le M\}.$$

Set the subintervals

$$\sigma_n = [x_n; x_{n+1}), n = 0,1,...,N-2; \quad \sigma_{N-1} = [x_{N-1}, x_N],$$

$$\delta_m = [y_m; y_{m+1}), m = 0,1,...,M-2; \quad \delta_{M-1} = [y_{M-1}, y_M],$$

and $D_{n,m} := \sigma_n \times \delta_m$ for all $n = 0,1,...,N-1; m = 0,1,...,M-1$.

Moreover, denote by $\pi_{p-1,p-1}$ the set of all real polynomials of degree not exceeding $p-1$ in x and y. We define the real polynomial spline space of degree $p-1$ in x and y as follows :

$$S^{(-1)}_{p-1,p-1}(\Pi_{N,M}) = \{v : v_{n,m} = v|_{D_{n,m}} \in \pi_{p-1,p-1}, n = 0, ..., N-1; m = 0, 1, ..., M-1\}. \quad (2.1.2)$$

This is the space of bivariate polynomial spline functions of degree (at most) $p-1$ in x and y. Its dimension is NMp^2, i.e., the same as the total number of the coefficients of the polynomials $v_{n,m}, n = 0, ..., N-1; m = 0, 1, ..., M-1$. To find these coefficients, we use Taylor polynomial on each rectangle. Note that the solution u of (2.1.1) is known on part of the boundary of D :

$u(x, y) = g(x, 0)$ if $0 \leq x \leq a$ and $y = 0$,

$u(x, y) = g(0, y)$ if $0 \leq y \leq b$ and $x = 0$.

First, we approximate u in the rectangle $D_{0,0}$ by the polynomial

$$v_{0,0}(x, y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} u(0, 0)}{\partial x^i \partial y^j} x^i y^j \; ; \quad (x, y) \in D_{0,0}, \quad (2.1.3)$$

where $\frac{\partial^{i+j} u(0, 0)}{\partial x^i \partial y^j}$ is the exact value of $\frac{\partial^{i+j} u}{\partial x^i \partial y^j}$ at point $(0, 0)$.

To find $\frac{\partial^j u(x, y)}{\partial y^j}$, we differentiate equation (2.1.1) j-times with respect to y

$$\begin{aligned}
\frac{\partial^j u(x, y)}{\partial y^j} &= \partial_2^{(j)} g(x, y) + \int_0^x \sum_{r=0}^{j-1} \frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)} K(x, y, t, y) u(t, y)] dt \\
&\quad + \int_0^x \int_0^y \partial_2^{(j)} K(x, y, t, s) u(t, s) ds dt \\
&= \partial_2^{(j)} g(x, y) + \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_0^x \frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} K(x, y, t, y)] \frac{\partial^l u(t, y)}{\partial y^l} dt \\
&\quad + \int_0^x \int_0^y \partial_2^{(j)} K(x, y, t, s) u(t, s) ds dt. \quad (2.1.4)
\end{aligned}$$

Now, we differentiate equation (2.1.4) i-times with respect to x, we obtain

$$\frac{\partial^{i+j} u(x, y)}{\partial x^i \partial y^j} = \partial_1^{(i)} \partial_2^{(j)} g(x, y)$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\frac{\partial^l u(x,y)}{\partial y^l}\right]$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l u(t,y)}{\partial y^l}dt$$
$$+\int_0^y\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)u(x,s)\right]ds$$
$$+\int_0^x\int_0^y\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)u(t,s)dsdt,$$

which implies,

$$\frac{\partial^{i+j}u(x,y)}{\partial x^i\partial y^j}=\partial_1^{(i)}\partial_2^{(j)}g(x,y)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}u(x,y)}{\partial x^\eta\partial y^l}$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l u(t,y)}{\partial y^l}dt$$
$$+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^\eta u(x,s)}{\partial x^\eta}ds$$
$$+\int_0^x\int_0^y\partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)u(t,s)dsdt.$$

Hence,

$$\frac{\partial^{i+j}u(0,0)}{\partial x^i\partial y^j}=\partial_1^{(i)}\partial_2^{(j)}g(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=y=0}\frac{\partial^{l+\eta}u(0,0)}{\partial x^\eta\partial y^l}.$$

Second, we approximate u in the rectangles $D_{n,0}, n=1,...,N-1$ by the polynomials

$$v_{n,0}(x,y)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,0}(x_n,0)}{\partial x^i\partial y^j}(x-x_n)^i y^j\ ;\quad (x,y)\in D_{n,0},\qquad(2.1.5)$$

where $\hat{v}_{n,0}$ is the exact solution of the integral equation :

$$\begin{aligned}\hat{v}_{n,0}(x,y) = g(x,y) &+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y K(x,y,t,s)v_{\xi,0}(t,s)dsdt \\ &+ \int_{x_n}^{x}\int_0^y K(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt.\end{aligned} \tag{2.1.6}$$

To find $\dfrac{\partial^j \hat{v}_{n,0}(x,y)}{\partial y^j}$, we differentiate equation (2.1.6) j-times with respect to y

$$\begin{aligned}\frac{\partial^j \hat{v}_{n,0}(x,y)}{\partial y^j} &= \partial_2^{(j)} g(x,y) + \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}K(x,y,t,y)v_{\xi,0}(t,y)]dt \\ &\quad + \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \partial_2^{(j)}K(x,y,t,s)v_{\xi,0}(t,s)dsdt \\ &\quad + \int_{x_n}^{x}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}[\partial_2^{(j-1-r)}K(x,y,t,y)\hat{v}_{n,0}(t,y)]dt \\ &\quad + \int_{x_n}^{x}\int_0^y \partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt \\ &= \partial_2^{(j)} g(x,y) + \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l v_{\xi,0}(t,y)}{\partial y^l}dt \\ &\quad + \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \partial_2^{(j)}K(x,y,t,s)v_{\xi,0}(t,s)dsdt \\ &\quad + \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\frac{\partial^l \hat{v}_{n,0}(t,y)}{\partial y^l}dt \\ &\quad + \int_{x_n}^{x}\int_0^y \partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt.\end{aligned} \tag{2.1.7}$$

Now, we differentiate equation (2.1.7) i-times with respect to x, we obtain

$$\begin{aligned}\frac{\partial^{i+j} \hat{v}_{n,0}(x,y)}{\partial x^i \partial y^j} =& \partial_1^{(i)}\partial_2^{(j)} g(x,y) \\ &+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l v_{\xi,0}(t,y)}{\partial y^l}dt\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\xi,0}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\frac{\partial^l\hat{v}_{n,0}(x,y)}{\partial y^l}\right]\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l\hat{v}_{n,0}(t,y)}{\partial y^l}dt\\
&+\int_0^y\sum_{q=0}^{i-1}\frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\hat{v}_{n,0}(x,s)\right]ds\\
&+\int_{x_n}^{x}\int_0^y \partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
&\frac{\partial^{i+j}\hat{v}_{n,0}(x,y)}{\partial x^i\partial y^j}=\partial_1^{(i)}\partial_2^{(j)}g(x,y)\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l v_{\xi,0}(t,y)}{\partial y^l}dt\\
&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)v_{\xi,0}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\hat{v}_{n,0}(x,y)}{\partial x^\eta\partial y^l}\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l\hat{v}_{n,0}(t,y)}{\partial y^l}dt\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^\eta\hat{v}_{n,0}(x,s)}{\partial x^\eta}ds\\
&+\int_{x_n}^{x}\int_0^y \partial_1^{(i)}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,0}(t,s)dsdt.
\end{aligned}
\tag{2.1.8}
$$

Hence,

$$
\frac{\partial^{i+j}\hat{v}_{n,0}(x_n,0)}{\partial x^i\partial y^j}=\partial_1^{(i)}\partial_2^{(j)}g(x_n,0)
$$

$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]_{x=x_n,y=0}\frac{\partial^l v_{\xi,0}(t,0)}{\partial y^l}dt$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=x_n,y=0}\frac{\partial^{l+\eta}\hat{v}_{n,0}(x_n,0)}{\partial x^\eta\partial y^l}.$$

Third, we approximate u by $v_{n,m}$ in the rectangles $D_{n,m}$, $n=0,...,N-1$ and $m=1,...,M-1$ such that,

$$v_{n,m}(x,y)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,m}(x_n,y_m)}{\partial x^i\partial y^j}(x-x_n)^i(y-y_m)^j\ ;\quad (x,y)\in D_{n,m}, \tag{2.1.9}$$

where $\hat{v}_{n,m}$ is the exact solution of the integral equation :

$$\begin{aligned}\hat{v}_{n,m}(x,y)=g(x,y)&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}}K(x,y,t,s)v_{\xi,\rho}(t,s)dsdt\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y}K(x,y,t,s)v_{\xi,m}(t,s)dsdt\\&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}K(x,y,t,s)v_{n,\rho}(t,s)dsdt\\&+\int_{x_n}^{x}\int_{y_m}^{y}K(x,y,t,s)\hat{v}_{n,m}(t,s)dsdt.\end{aligned}\tag{2.1.10}$$

To find $\dfrac{\partial^j\hat{v}_{n,m}(x,y)}{\partial y^j}$, we differentiate equation (2.1.10) j-times with respect to y

$$\begin{aligned}\frac{\partial^j\hat{v}_{n,m}(x,y)}{\partial y^j}=&\partial_2^{(j)}g(x,y)+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_2^{(j)}K(x,y,t,s)v_{\xi,\rho}(t,s)dsdt\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}\left[\partial_2^{(j-1-r)}K(x,y,t,y)v_{\xi,m}(t,y)\right]dt\\&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y}\partial_2^{(j)}K(x,y,t,s)v_{\xi,m}(t,s)dsdt\\&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}\partial_2^{(j)}K(x,y,t,s)v_{n,\rho}(t,s)dsdt\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_n}^{x}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y^r}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\hat{v}_{n,m}(t,y)\right]dt\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,m}(t,s)dsdt\\
=&\partial_2^{(j)}g(x,y)+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_2^{(j)}K(x,y,t,s)v_{\xi,\rho}(t,s)dsdt\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\frac{\partial^l v_{\xi,m}(t,y)}{\partial y^l}dt\\
&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y}\partial_2^{(j)}K(x,y,t,s)v_{\xi,m}(t,s)dsdt\\
&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}\partial_2^{(j)}K(x,y,t,s)v_{n,\rho}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\frac{\partial^l v_{\xi,m}(t,y)}{\partial y^l}dt\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_2^{(j)}K(x,y,t,s)\hat{v}_{n,m}(t,s)dsdt. \qquad (2.1.11)
\end{aligned}
$$

Now, we differentiate equation (2.1.11) i-times with respect to x, we obtain

$$
\begin{aligned}
\frac{\partial^{i+j}\hat{v}_{n,m}(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i)}\partial_2^{(j)} g(x,y) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{\xi,\rho}(t,s)\,dsdt \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l v_{\xi,m}(t,y)}{\partial y^l}dt \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{\xi,m}(t,s)\,dsdt \\
&+ \sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\int_{y_\rho}^{y_{\rho+1}} \frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)v_{n,\rho}(x,s)\right]ds \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{n,\rho}(t,s)\,dsdt \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\binom{r}{l}\frac{\partial^q}{\partial x^q}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\frac{\partial^l \hat{v}_{n,m}(x,y)}{\partial y^l}\right] \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l \hat{v}_{n,m}(t,y)}{\partial y^l}dt \\
&+ \sum_{q=0}^{i-1}\int_{y_m}^{y} \frac{\partial^q}{\partial x^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\hat{v}_{n,m}(x,s)\right]ds \\
&+ \int_{x_n}^{x}\int_{y_m}^{y} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s)\hat{v}_{n,m}(t,s)\,dsdt,
\end{aligned}
$$

which implies,

$$\begin{aligned}
\frac{\partial^{i+j}\hat{v}_{n,m}(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i)}\partial_2^{(j)} g(x,y) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{\xi,\rho}(t,s) ds dt \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} K(x,y,t,y)]\right]\frac{\partial^l v_{\xi,m}(t,y)}{\partial y^l} dt \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{\xi,m}(t,s) ds dt \\
&+ \sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} K(x,y,x,s)\right]\frac{\partial^\eta v_{n,\rho}(x,s)}{\partial x^\eta} ds \\
&+ \sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s) v_{n,\rho}(t,s) ds dt \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\hat{v}_{n,m}(x,y)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} K(x,y,t,y)]\right]\frac{\partial^l \hat{v}_{n,m}(t,y)}{\partial y^l} dt \\
&+ \sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_m}^{y} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} K(x,y,x,s)\right]\frac{\partial^\eta \hat{v}_{n,m}(x,s)}{\partial x^\eta} ds \\
&+ \int_{x_n}^{x}\int_{y_m}^{y} \partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s)\hat{v}_{n,m}(t,s) ds dt.
\end{aligned} \tag{2.1.12}$$

Hence,

$$\begin{aligned}
\frac{\partial^{i+j}\hat{v}_{n,m}(x_n,y_m)}{\partial x^i \partial y^j} &= \partial_1^{(i)}\partial_2^{(j)} g(x_n,y_m) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} K(x_n,y_m,t,s) v_{\xi,\rho}(t,s) ds dt \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} K(x,y,t,y)]\right]_{x=x_n,y=y_m}\frac{\partial^l v_{\xi,m}(t,y_m)}{\partial y^l} dt \\
&+ \sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} K(x,y,x,s)\right]_{x=x_n,y=y_m}\frac{\partial^\eta v_{n,\rho}(x_n,s)}{\partial x^\eta} ds \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} K(x,y,t,y)\right]\right)\right]_{x=x_n,y=y_m}\frac{\partial^{l+\eta}\hat{v}_{n,m}(x_n,y_m)}{\partial x^\eta \partial y^l},
\end{aligned}$$

for $n = 0, ..., N-1$ and $m = 1, ..., M-1$.

2.2 Convergence analysis

We consider the space $L^\infty(D)$ with the norm

$$\left\|\varphi\right\|_{L^\infty(D)} = \inf\left\{C \in \mathbb{R} : \left|\varphi(x,y)\right| \leq C \text{ for a.e. } (x,y) \in D\right\} < \infty.$$

We need the following lemma to prove the convergence of the presented method.

Lemma 2.2.1 *Let g and K be p times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(p)$ such that for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$ and $i + j = 0, 1, ..., p$, we have,*

$$\left\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i \partial y^j}\right\|_{L^\infty(D_{n,m})} \leq \alpha(p),$$

where $\hat{v}_{0,0}(x,y) = u(x,y)$ for $(x,y) \in D_{0,0}$.

Proof. Let $a^{i,j}_{n,m} = \|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i \partial y^j}\|_{L^\infty(D_{n,m})}$, we have for all $i + j = 0, 1, ..., p$,

$$a^{i,j}_{0,0} \leq \max\left\{\left\|\frac{\partial^{i+j}u}{\partial x^i \partial y^j}\right\|_{L^\infty(D_{0,0})}, i + j = 0, 1, ..., p\right\} = \alpha_1(p). \tag{2.2.1}$$

Now, we consider the sequence $\Gamma_n = \max\{a^{i,j}_{n,0}, i + j = 0, \ldots, p\}$, $n = 0, 1, ..., N-1$.
From (2.1.8), we have for all $n = 1, \ldots, N-1$ and $i + j = 0, 1, ..., p$

$$\begin{aligned} a^{i,j}_{n,0} \leq{}& c_1 + c_2 \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{x_\xi}^{x_{\xi+1}} \Gamma_\xi dt + c_3 \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \Gamma_\xi ds dt + c_4 \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} a^{\eta,l}_{n,0} \\ &+ c_2 \sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{x_n}^{x} a^{0,l}_{n,0} dt + c_5 \int_0^y \sum_{q=0}^{i-1}\sum_{\eta=0}^{q} a^{\eta,0}_{n,0} ds + c_3 \int_{x_n}^{x}\int_0^y a^{0,0}_{n,0} ds dt, \end{aligned}$$

where, for all $i + j = 0, 1, ..., p$

$$c_1 = \max\left\{\left\|\partial_1^{(i)}\partial_2^{(j)} g\right\|_{L^\infty(D)}\right\},$$

$$c_2 = \max\left\{\binom{r}{l}\left\|\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\right\|_{L^\infty(D)}, r = 0,...,j-1; l = 0,...,r\right\},$$

$$c_3 = \max\left\{\left\|\partial_1^{(i)}\partial_2^{(j)} K(x,y,t,s)\right\|_{L^\infty(D)}\right\},$$

$$c_4 = \max\left\{\begin{array}{c}\binom{r}{l}\binom{q}{\eta}\left\|\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\right\|_{L^\infty(D)}, r = 0,...,j-1; \\ l = 0,...,r; q = 0,...,i-1; \eta = 0,...,q\end{array}\right\},$$

and

$$c_5 = \max\left\{\binom{q}{\eta}\left\|\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K(x,y,x,s)\right]\right\|_{L^\infty(D)}, q = 0,...,i-1; \eta = 0,...,q\right\},$$

the constants $c_i, i = 1,...,5$ are positive and independent of N and M, hence,

$$\begin{aligned} a_{n,0}^{i,j} \le c_1 + c_2p^2h\sum_{\xi=0}^{n-1}\Gamma_\xi + c_3hk\sum_{\xi=0}^{n-1}\Gamma_\xi + c_4\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} a_{n,0}^{\eta,l} \\ + c_2h\sum_{r=0}^{j-1}\sum_{l=0}^{r} a_{n,0}^{0,l} + c_5k\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} a_{n,0}^{\eta,0} + c_3hka_{n,0}^{0,0}, \end{aligned}$$

which implies that,

$$\begin{aligned} a_{n,0}^{i,j} &\le c_1 + c_2p^2h\sum_{\xi=0}^{n-1}\Gamma_\xi + c_3hk\sum_{\xi=0}^{n-1}\Gamma_\xi + c_4p^2\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l} + c_2ph\sum_{l=0}^{j-1} a_{n,0}^{0,l} \\ &\quad + c_5pk\sum_{\eta=0}^{i-1} a_{n,0}^{\eta,0} + c_3hka_{n,0}^{0,0} \qquad (2.2.2) \\ &\le c_1 + c_6h\sum_{\xi=0}^{n-1}\Gamma_\xi + c_7\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} a_{n,0}^{\eta,l}, \end{aligned}$$

where $c_6 = c_2p^2 + c_3k$ and $c_7 = c_4p^2 + c_2ph + c_5pk + c_3hk$.
Using the notations of Lemma 1.6.4, we put for each fixed $n \in \mathbb{N}$

$$\varepsilon(i,j) = a_{n,0}^{i,j}, \quad a(i,j) = c_1 + c_6h\sum_{\xi=0}^{n-1}\Gamma_\xi, \quad b(i,j) = c_7.$$

It is clear that the sequence $a(i,j)$ is nondecreasing in each variable i and j.

Then, by Lemma 1.6.4, we obtain from (2.2.2)

$$\begin{aligned} a_{n,0}^{i,j} &\le \left(c_1 + c_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi\right) \prod_{s=0}^{i-1} \left[1 + \sum_{t=0}^{j-1} c_7\right] \\ &\le \underbrace{c_1 \left[1 + pc_7\right]^p}_{c_8} + \underbrace{c_6 \left[1 + pc_7\right]^p}_{c_9} h \sum_{\xi=0}^{n-1} \Gamma_\xi, \end{aligned} \tag{2.2.3}$$

which implies that,

$$\Gamma_n \le c_8 + c_9 h \sum_{\xi=0}^{n-1} \Gamma_\xi. \tag{2.2.4}$$

It follows, by Lemma 1.6.2, for all $n = 0, 1, ..., N-1$

$$\Gamma_n \le c_8 \exp(c_9 a). \tag{2.2.5}$$

On the other hand, we have, from (2.1.12), for all $n = 0, ..., N-1$, $m = 1, ..., M-1$ and $i + j = 0, ..., p$

$$\begin{aligned} a_{n,m}^{i,j} \le c_1 &+ c_3 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a_{\xi,\rho}^{s,t} + c_2 h \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{s+t=0}^{p-1} a_{\xi,m}^{s,t} \\ &+ c_3 hk \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} a_{\xi,m}^{s,t} + c_5 k \sum_{\rho=0}^{m-1} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \sum_{s+t=0}^{p-1} a_{n,\rho}^{s,t} \\ &+ kc_3 h \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a_{n,\rho}^{s,t}(t,s) + c_4 \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a_{n,m}^{\eta,l} \\ &+ c_2 h \sum_{r=0}^{j-1} \sum_{l=0}^{r} a_{n,m}^{0,l} + c_5 k \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} a_{n,m}^{\eta,0} + kc_3 h a_{n,m}^{0,0} \\ \le c_1 &+ c_3 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a_{\xi,\rho}^{s,t} + c_2 p^2 h \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} a_{\xi,m}^{s,t} \\ &+ c_3 hk \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} a_{\xi,m}^{s,t} + c_5 p^2 k \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a_{n,\rho}^{s,t} \end{aligned}$$

$$+ hkc_3 \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} a_{n,\rho}^{s,t} + c_4 p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,m}^{\eta,l}$$

$$+ c_2 ph \sum_{l=0}^{j-1} a_{n,m}^{0,l} + c_5 pk \sum_{\eta=0}^{i-1} a_{n,m}^{\eta,0} + hkc_3 a_{n,m}^{0,0}. \tag{2.2.6}$$

Consider, the sequence $\Gamma_{n,m} = \max\{a_{n,m}^{i,j}, i+j = 0,\ldots,p\}, n = 0,1,...,N-1; m = 0,...,M-1$, then by (2.2.6), we have

$$a_{n,m}^{i,j} \leq c_1 + c_3 p^2 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + c_2 p^4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m}$$

$$+ c_3 p^2 hk \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + c_5 p^4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}$$

$$+ hkc_3 p^2 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + c_4 p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,m}^{\eta,l}$$

$$+ c_2 ph \sum_{l=0}^{j-1} a_{n,m}^{0,l} + c_5 pk \sum_{\eta=0}^{i-1} a_{n,m}^{\eta,0} + hkc_3 a_{n,m}^{0,0},$$

which implies that,

$$a_{n,m}^{i,j} \leq c_1 + b_1 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_2 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_3 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + b_4 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,m}^{\eta,l}, \tag{2.2.7}$$

where $b_1 = c_3 p^2$, $b_2 = c_2 p^4 + c_3 p^2 k$, $b_3 = c_5 p^4 + hc_3 p^2$ and $b_4 = c_4 p^2 + c_2 ph + c_5 pk + c_3 hk$. Using the notations of Lemma 1.6.4, we put for each fixed $n, m \in \mathbb{N}$

$$\varepsilon(i,j) = a_{n,m}^{i,j}, a(i,j) = c_1 + b_1 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_2 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_3 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, b(i,j) = b_4.$$

It is clear that the sequence $a(i, j)$ is nondecreasing in each variable i and j.

Then, by Lemma 1.6.4, we obtain from (2.2.7)

$$\begin{aligned} a_{n,m}^{i,j} \leq & \left(c_1 + b_1 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_2 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + b_3 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) \prod_{s=0}^{i-1} \left[1 + \sum_{t=0}^{j-1} b_4 \right] \\ & \leq \underbrace{c_1 \left[1 + pb_4\right]^p}_{b_5} + hk \underbrace{b_1 \left[1 + pb_4\right]^p}_{b_6} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + h \underbrace{b_2 \left[1 + pb_4\right]^p}_{b_7} \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ & \quad + k \underbrace{b_3 \left[1 + pb_4\right]^p}_{b_8} \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \end{aligned}$$

it follows that, for all $n = 0, 1, ..., N-1; m = 0, ..., M-1$,

$$\Gamma_{n,m} \leq b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + hb_7 \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}. \tag{2.2.8}$$

Using the notations of Lemma 1.6.3, we put for each fixed $m \in \mathbb{N}$

$$\varepsilon(n) = \Gamma_{n,m}, a(n) = b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, b(n) = hb_7.$$

Then, by Lemma 1.6.3, we obtain from (2.2.8)

$$\begin{aligned}
\Gamma_{n,m} \leq & b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \\
& + \sum_{s=0}^{n-1} \left(b_5 + hkb_6 \sum_{\xi=0}^{s-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{s,\rho} \right) hb_7 \prod_{\sigma=s+1}^{n-1} [1 + hb_7] \\
\leq & b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \\
& + \sum_{s=0}^{n-1} \left(b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{s,\rho} \right) hb_7 \exp(ab_7) \\
\leq & b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \\
& + ab_7 \exp(ab_7) \left(b_5 + hkb_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} \right) + hkb_7 \exp(ab_7)\, b_8 \sum_{s=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{s,\rho} \\
\leq & \underbrace{b_5(1 + ab_7 \exp(ab_7))}_{b_9} + hk \underbrace{(b_6 + (ab_6 + b_8)b_7 \exp(ab_7))}_{b_{10}} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} \\
& + kb_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}.
\end{aligned} \tag{2.2.9}$$

Again, using the notations of Lemma 1.6.3, we put for each fixed $n \in \mathbb{N}$

$$\varepsilon(m) = \Gamma_{n,m}, a(m) = b_9 + hkb_{10} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho}, b(m) = kb_8.$$

Then, by Lemma 1.6.3, we obtain from (2.2.9)

$$\begin{aligned}\Gamma_{n,m} \leq & b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \sum_{s=0}^{m-1}\left(b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{s-1}\Gamma_{\xi,\rho}\right)kb_8\prod_{\sigma=s+1}^{m-1}\left[1+kb_8\right]\\
\leq & b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \sum_{s=0}^{m-1}\left(b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho}\right)kb_8\exp\left(bb_8\right)\\
\leq & b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \left(b_9 + hkb_{10}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho}\right)bb_8\exp\left(bb_8\right)\\
\leq & \underbrace{b_9\left(1+bb_8\exp\left(bb_8\right)\right)}_{b_{11}} + hk\underbrace{b_{10}\left(1+bb_8\exp\left(bb_8\right)\right)}_{b_{12}}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho}.\end{aligned}$$

Then, by Lemma 1.6.4, we have

$$\Gamma_{n,m} \leq b_{11}\prod_{\xi=0}^{n-1}\left[1+\sum_{\rho=0}^{m-1}hkb_{12}\right] \leq b_{11}\exp\left(abb_{12}\right). \qquad (2.2.10)$$

Hence, from (2.2.1), (2.2.5) and (2.2.10), by setting

$$\alpha(p) = \max\left\{\alpha_1(p), c_{10}\exp(pc_{11}a), b_{11}\exp\left(abb_{12}\right)\right\}.$$

The proof of Lemma 2.2.1 is completed. ■

The following theorem gives the convergence of the presented method.

Theorem 2.2.1 *Let g and K be p times continuously differentiable on their respective domains. Then equations* (2.1.3),(2.1.5),(2.1.9) *define a unique approximation $v \in S^{(-1)}_{p-1,p-1}(\Pi_{N,M})$, and the resulting error function $e(x,y) = u(x,y) - v(x,y)$ satisfies:*

$$\|e\|_{L^\infty(D)} \leq C(h+k)^p,$$

where C is a finite constant independent of h.

Proof. Define the error $e(x,y)$ on $D_{n,m}$ by $e_{n,m}(x,y) = u(x,y) - v_{n,m}(x,y)$ for all $n \in \{0,\dots,N-1\}$ and $m \in \{0,\dots,M-1\}$.
The proof is split into three steps.
Claim 1. There exists a constant C_1 independent of h and k such that,

$$\|e_{0,0}\|_{L^\infty(D_{0,0})} \le C_1(h+k)^p.$$

Let $(x,y) \in D_{0,0}$, by using Lemma 1.1.1, we obtain from (2.1.3)

$$|e_{0,0}(x,y)| \le \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\| h^i k^j.$$

Hence, by Lemma 2.2.1, we have

$$|e_{0,0}(x,y)| \le \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \underbrace{\frac{\alpha(p)}{p!}}_{C_1} (h+k)^p. \tag{2.2.11}$$

Claim 2. There exists a constant C_2 independent of h and k such that,

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \le C_2(h+k)^p,$$

for all $n = 1,\dots,N-1$. Let $(x,y) \in D_{n,0}$, we have from (2.1.6)

$$\begin{aligned} u(x,y) - \hat{v}_{n,0}(x,y) = & \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_0^y K(x,y,t,s) e_{\xi,0}(t,s)\,ds\,dt \\ & + \int_{x_n}^{x} \int_0^y K(x,y,t,s)(u(t,s) - \hat{v}_{n,0}(t,s))\,ds\,dt. \end{aligned}$$

Hence,

$$|u(x,y) - \hat{v}_{n,0}(x,y)| \le \sum_{\xi=0}^{n-1} hk\overline{K} \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \overline{K} \int_{x_n}^{x} \int_0^y |u(t,s) - \hat{v}_{n,0}(t,s)|\,ds\,dt,$$

where, $\overline{K} = max\{||K||_{L^{\infty}(D)}\}$.
Then by Lemma 1.6.1,

$$\begin{aligned}|u(x,y) - \hat{v}_{n,0}(x,y)| &\leq \sum_{\xi=0}^{n-1} hk\overline{K}||e_{\xi,0}||_{L^{\infty}(D_{\xi,0})} \exp\left(\overline{K}ab\right)\\ &\leq \sum_{\xi=0}^{n-1} h \underbrace{b\overline{K}\exp\left(\overline{K}ab\right)}_{d_1} ||e_{\xi,0}||_{L^{\infty}(D_{\xi,0})}.\end{aligned}$$

Which implies, by using Lemma 1.1.1, that

$$\begin{aligned}||e_{n,0}||_{L^{\infty}(D_{n,0})} &\leq ||u - \hat{v}_{n,0}|| + ||\hat{v}_{n,0} - v_{n,0}||\\ &\leq \sum_{\xi=0}^{n-1} hd_1||e_{\xi,0}||_{L^{\infty}(D_{\xi,0})} + \sum_{i+j=p} \frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{n,0}}{\partial x^i \partial y^j}\right\| h^i k^j.\end{aligned}$$

Hence, by Lemma 2.2.1, we obtain

$$||e_{n,0}||_{L^{\infty}(D_{n,0})} \leq \sum_{\xi=0}^{n-1} hd_1||e_{\xi,0}||_{L^{\infty}(D_{\xi,0})} + \frac{\alpha(p)}{p!}(h+k)^p.$$

Then, by Lemma 1.6.2, we have

$$||e_{n,0}||_{L^{\infty}(D_{n,0})} \leq \frac{\alpha(p)}{p!}(h+k)^p \exp(ad_1).$$

Thus, we take $C_2 = \frac{\alpha(p)}{p!}\exp(ad_1)$.
Claim 3. There exists a constant C_3 independent of h and k such that,

$$||e_{n,m}||_{L^{\infty}(D_{n,m})} \leq C_3(h+k)^p,$$

for all $n = 0, \dots, N-1$ and $m = 1, \dots, M-1$. Let $(x, y) \in D_{n,m}$, we have from (2.1.10)

$$|u(x,y) - \hat{v}_{n,m}(x,y)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{K}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\overline{K}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\overline{K}\|e_{n,\rho}\|$$
$$+ \overline{K}\int_{x_n}^{x}\int_{y_m}^{y} |u(t,s) - \hat{v}_{n,m}(t,s)|dsdt.$$

Then by Lemma 1.6.1,

$$|u(x,y) - \hat{v}_{n,m}(x,y)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\underbrace{\overline{K}\exp\left(\overline{K}ab\right)}_{d_2}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\underbrace{\overline{K}\exp\left(\overline{K}ab\right)}_{d_2}\|e_{\xi,m}\|$$
$$+ \sum_{\rho=0}^{m-1} hk\underbrace{\overline{K}\exp\left(\overline{K}ab\right)}_{d_2}\|e_{n,\rho}\|,$$

which implies, by using Lemma 1.1.1, that

$$\begin{aligned}\|e_{n,m}\|_{L^\infty(D_{n,0})} \leq& \|u - \hat{v}_{n,m}\| + \|\hat{v}_{n,m} - v_{n,m}\| \\ \leq& \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkd_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| \\ &+ \sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial x^i\partial y^j}\right\| h^i k^j,\end{aligned}$$

Hence, by Lemma 2.2.1, we obtain

$$\|e_{n,m}\| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkd_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p. \tag{2.2.12}$$

Using the notations of Lemma 1.6.3, we put for each fixed $m \in \mathbb{N}$

$$\varepsilon(n) = \|e_{n,m}\|, a(n) = \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p, b(n) = hkd_2.$$

Then, by Lemma 1.6.3, we obtain from (2.2.12)

$$\begin{aligned}
\|e_{n,m}\| &\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p \\
&\quad + \sum_{s=0}^{n-1}\left(\sum_{\xi=0}^{s-1}\sum_{\rho=0}^{m-1} hkd_2\|e_{\xi,\rho}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{s,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p\right) hkd_2 \exp(abd_2) \\
&\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\,\underbrace{d_2\left(1+2abd_2\exp(abd_2)\right)}_{d_3}\|e_{\xi,\rho}\| + \sum_{\rho=0}^{m-1} hkd_2\|e_{n,\rho}\| \\
&\quad + \underbrace{\frac{\alpha(p)}{p!}\left(1+abd_2\exp(abd_2)\right)}_{d_4}(h+k)^p.
\end{aligned} \tag{2.2.13}$$

Again, using the notations of Lemma 1.6.3, we put for each fixed $n \in \mathbb{N}$

$$\varepsilon(m) = \|e_{n,m}\|, a(m) = \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_3\|e_{\xi,\rho}\| + d_4(h+k)^p, b(m) = hkd_2.$$

Then, by Lemma 1.6.3, we obtain from (2.2.13)

$$\begin{aligned}
\|e_{n,m}\| &\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkd_3\|e_{\xi,\rho}\| + d_4(h+k)^p \\
&\quad + \sum_{s=0}^{m-1}\left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{s-1} hkd_3\|e_{\xi,\rho}\| + d_4(h+k)^p\right) hkd_2 \exp(abd_2) \\
&\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\,\underbrace{d_3\left(1+abd_2\exp(abd_2)\right)}_{d_5}\|e_{\xi,\rho}\| + \underbrace{d_4\left(1+abd_2\exp(abd_2)\right)}_{d_6}(h+k)^p.
\end{aligned}$$

Moreover, by Lemma (1.6.4), we deduce that

$$\|e_{n,m}\| \le d_6(h+k)^p \prod_{s=0}^{n-1}\left[1+\sum_{t=0}^{m-1} hkd_5\right] \le \underbrace{d_6\exp(abd_5)}_{C_3}(h+k)^p.$$

Thus, the proof is completed by taking $C = \max\{C_1, C_2, C_3\}$. ■

CHAPTER 3

NUMERICAL EXPERIMENTS

In this chapter, we present three examples to show the performance of the described method in the previous chapter for solving the integral equation (2.1.1). We calculate the error between u and the Taylor collocation solution v, all the exact solutions u are already known. Moreover, we compare our results with other methods : two-dimensional orthogonal triangular functions [1], and the differential transform method [2]. All the calculations are supported by the software Maple17.

3.1 Illustrative example

Example 3.1.1 *Consider the following linear 2D-VIEs [13]*

$$u(x, y) = g(x, y) + \int_0^x \int_0^y (xt^2 + \cos(s))u(t, s)dsdt,$$

where $(x, y) \in [0, 1] \times [0, 1]$, *and* $g(x, y) = x\sin(y) + \frac{x^5}{4}(\cos(y) - 1) - \frac{x^2}{4}\sin^2(y)$ *with the exact solution is* $u(x, y) = x\sin(y)$.

Applying Taylor collocation method on the above integral equation, we approximate the solution on $D_{n,m}$ *for* $n = 0, 1, ..., N-1; m = 0, 1, ..., M-1$ *by the Taylor collocation polynomial* $v_{n,m}$ *shown some of them in the Table 3.1 for* $N = 10, M = 20$, *and* $p = 3$.

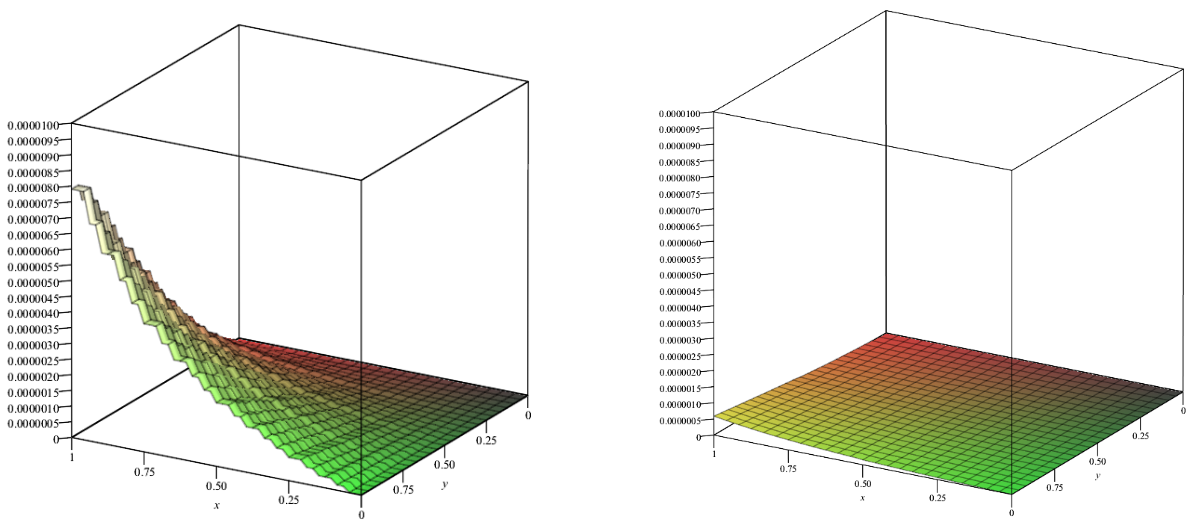
Table 3.2 and Table 3.4 show the absolute errors at some points with $p = 3$ *and* $(N, M) \in \{(10, 10), (20, 20), (50, 50), (10, 20)\}$, *Table 3.3 shows the absolute errors at some points with* $p = 4, 5, 6$ *and* $(N, M) = (20, 20), (50, 50), (100, 100)$ *respectively.*

The absolute errors function for $p = 3$ *and* $(N, M) \in \{(20, 20), (50, 50), (10, 20)\}$ *are plotted in Fig. 3.1 and Fig. 3.3 , we note that the absolute error reduces as N and M increases. As well as comparing the absolute errors function for* $p = 3, 4$ *and* $N = M = 30$ *are plotted in Fig. 3.2, we note that the absolute error reduces as p increases.*

The results in this example confirm the theoretical results (see, for example, the column 4 in Table 3.2 and Table 3.3). Moreover, the absolute error decreases as N, M *or* p *increase.*

Table 3.1: A some of Taylor collocation polynomials of Example 3.1.1

Domain $D_{n,m}$	Taylor collocation polynomial $v_{n,m}$
$D_{0,0}$	$v_{0,0}(x,y) = xy$
$D_{1,1}$	$v_{1,1}(x,y) = -0.000006247388393 + 0.0002498957665y + 0.00004165610x$ $-0.00249895846460505y^2 + 0.998750261602268xy$ $+1.30760740500000 \times 10^{-7}x^2$
$D_{1,15}$	$v_{1,15}(x,y) = -0.01917018430 + 0.05112289198y + 0.1328629567x$ $-0.0340819432792932y^2 + 0.731689211700456xy$ $0.000046586187771000x^2$
$D_{5,5}$	$v_{5,5}(x,y) = -0003863895322 + 0.03092534094y + 0.0051696897x$ $-0.0618509940890720y^2 + 0.968912917693200xy$ $0.0000068375131365000x^2$
$D_{5,15}$	$v_{5,15}(x,y) = -0.09583972142 + 0.2556140081y + 0.1328180967x$ $-0.170409873720890y^2 + 0.731691468236635xy$ $0.000054953918703500x^2$
$D_{9,9}$	$v_{9,9}(x,y) = -0.03960509782 + 0.1761570265y + 0.0296988148x$ $-0.195734518081094y^2 + 0.900453209408124xy$ $0.000037792320555000x^2$
$D_{9,19}$	$v_{9,19}(x,y) = -0.3302234215 + 0.6954611277y + 0.2605544964x$ $-0.366038085466712y^2 + 0.581699623103681xy$ $+0.00014543038420000x^2$



(a) The function e for N = M = 20.

(b) The function e for N = M = 50.

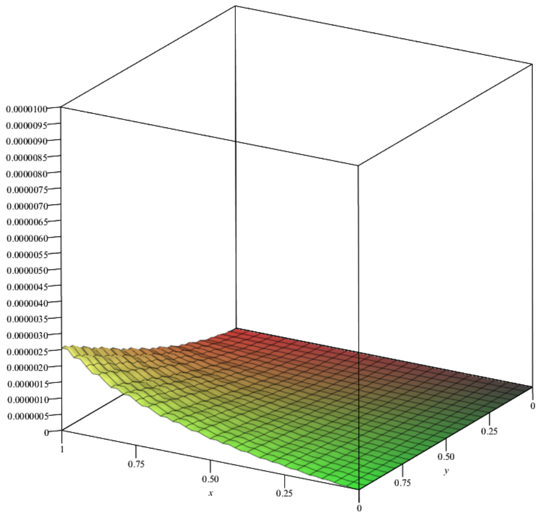
Figure 3.1: Plot of the absolute errors function of Example 3.1.1 for $p = 3$

Table 3.2: Absolute errors of Example 3.1.1 for $p = 3$ and $N = M = 10, 20, 50$

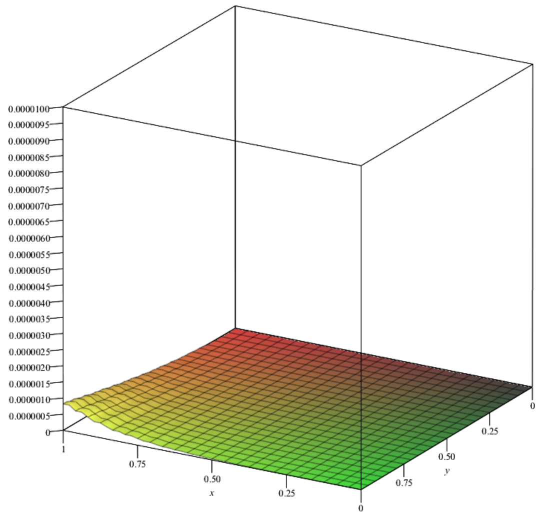
(x, y)	$N = M = 10$	$N = M = 20$	$N = M = 50$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$2.07e-08$	$5.19e-09$	$4.32e-10$
$(0.2, 0.2)$	$3.29e-07$	$5.16e-08$	$3.70e-09$
$(0.3, 0.3)$	$1.29e-06$	$1.85e-07$	$1.28e-08$
$(0.4, 0.4)$	$3.28e-06$	$4.54e-07$	$3.06e-08$
$(0.5, 0.5)$	$6.71e-06$	$9.11e-07$	$6.11e-08$
$(0.6, 0.6)$	$1.21e-05$	$1.62e-06$	$1.08e-07$
$(0.7, 0.7)$	$2.01e-05$	$2.68e-06$	$1.77e-07$
$(0.8, 0.8)$	$3.19e-05$	$4.21e-06$	$2.78e-07$
$(0.9, 0.9)$	$4.89e-05$	$6.43e-06$	$4.24e-07$

Table 3.3: Absolute errors of Example 3.1.1for $p = 4, 5, 6$ and $N = M = 20, 50, 100$

(x, y)	$p = 4, N = M = 20$	$p = 5, N = M = 50$	$p = 6, N = M = 100$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$1.29e-09$	$1.17e-11$	$1.10e-12$
$(0.2, 0.2)$	$5.23e-09$	$1.25e-11$	$8.20e-12$
$(0.3, 0.3)$	$1.26e-08$	$1.60e-10$	$3.09e-11$
$(0.4, 0.4)$	$2.77e-08$	$8.52e-10$	$1.94e-12$
$(0.5, 0.5)$	$6.26e-08$	$2.56e-09$	$2.75e-10$
$(0.6, 0.6)$	$1.45e-07$	$8.36e-09$	$8.35e-10$
$(0.7, 0.7)$	$3.37e-07$	$2.18e-08$	$2.91e-09$
$(0.8, 0.8)$	$7.57e-07$	$5.04e-08$	$4.75e-09$
$(0.9, 0.9)$	$1.63e-06$	$1.14e-07$	$1.30e-08$



(c) The function e for $p = 3$.



(d) The function e for $p = 4$.

Figure 3.2: Plot of the absolute errors function of Example 3.1.1 for $N = M = 30$

Table 3.4: Absolute errors of Example 3.1.1 for $p = 3$ and $N = 10, M = 20$

(x, y)	$N = 10, M = 20$	(x, y)	$N = 10, M = 20$
$(0.0, 0.0)$	0	$(0.1, 0.5)$	$2.27e - 07$
$(0.1, 0.05)$	$1.30e - 09$	$(0.2, 0.5)$	$4.84e - 07$
$(0.2, 0.1)$	$2.08e - 08$	$(0.3, 0.5)$	$7.73e - 07$
$(0.3, 0.15)$	$8.26e - 08$	$(0.4, 0.7)$	$1.96e - 06$
$(0.4, 0.2)$	$2.12e - 07$	$(0.5, 0.7)$	$2.63e - 06$
$(0.5, 0.25)$	$4.42e - 07$	$(0.6, 0.7)$	$3.42e - 06$
$(0.6, 0.3)$	$8.10e - 07$	$(0.7, 0.7)$	$4.38e - 06$
$(0.7, 0.35)$	$1.37e - 06$	$(0.8, 0.9)$	$8.21e - 06$
$(0.8, 0.4)$	$2.21e - 06$	$(0.9, 0.9)$	$1.04e - 05$
$(0.9, 0.45)$	$3.43e - 06$	$(1, 0.5)$	$7.74e - 05$

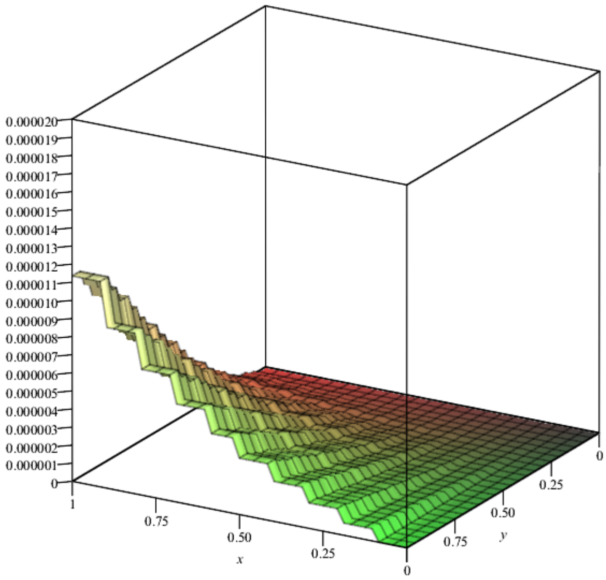


Figure 3.3: Plot of the absolute errors function of Example 3.1.1 for $p = 3, N = 10, M = 20$

3.2 Comparison examples

Example 3.2.1 *Consider the following linear 2D-VIEs [2]*

$$u(x,y) = g(x,y) + \int_0^x \int_0^y \frac{2}{e^{s-t}} u(t,s)dsdt, \qquad (x,y) \in [0,1] \times [0,1],$$

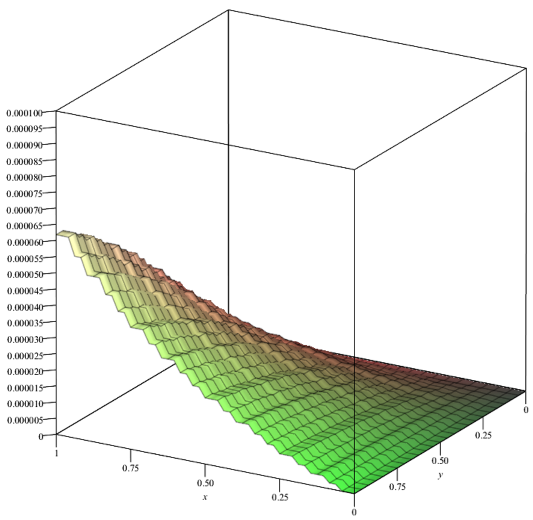
where $g(x,y) = (e^{x-y}+1)\sin(x+y) - e^{-y}\sin(y) - e^x \sin(x)$ *is chosen so that the exact solution is* $u(x,y) = \sin(x+y)$.

Table 3.5 shows the absolute errors at some points with $p = 3,4$ *and* $(N,M) \in \{(10,10),(20,20)\}$. *The absolute errors function for* $p = 3$ *and* $(N,M) \in \{(20,20),(40,40)\}$ *are plotted in Fig. 3.4.*

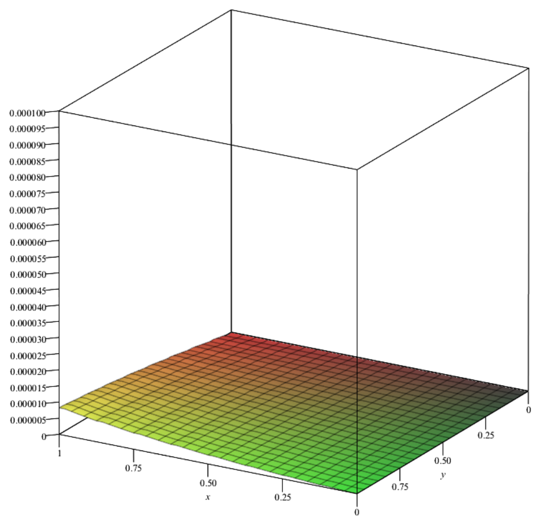
The numerical results for $p = 4$ *and* $h = k = 0.1$ *are compared with the numerical results obtained by using the differential transform method (DTM) [2] in Table 3.6. We note that the proposed method approximates the solution of the Volterra integral equation with a kernel of a* *general form but the results obtained are consistent with those obtained in [2].*

Table 3.5: Absolute errors of Example 3.2.1 for $p = 3, 4$ and $N = M = 10, 20$

(x, y)	$p = 3$ $N = M = 10$	$p = 3$ $N = M = 20$	$p = 4$ $N = M = 10$	$p = 4$ $N = M = 20$
$(0, 0)$	0	0	0	0
$(0.1, 0.1)$	$4.99e-06$	$6.25e-07$	$5.01e-09$	$2.03e-08$
$(0.2, 0.2)$	$2.00e-05$	$2.50e-06$	$1.26e-06$	$2.47e-07$
$(0.3, 0.3)$	$4.50e-05$	$5.63e-06$	$5.86e-06$	$9.48e-07$
$(0.4, 0.4)$	$8.02e-05$	$1.00e-05$	$1.61e-05$	$2.43e-06$
$(0.5, 0.5)$	$1.25e-04$	$1.57e-05$	$3.50e-05$	$5.08e-06$
$(0.6, 0.6)$	$1.82e-04$	$2.28e-05$	$6.62e-05$	$9.40e-06$
$(0.7, 0.7)$	$2.51e-04$	$3.14e-05$	$1.14e-04$	$1.60e-05$
$(0.8, 0.8)$	$3.34e-04$	$4.18e-05$	$1.88e-04$	$2.61e-05$
$(0.9, 0.9)$	$4.35e-04$	$5.44e-05$	$2.97e-04$	$4.08e-05$
$(1.0, 1.0)$	$1.77e-04$	$1.14e-05$	$1.35e-03$	$1.76e-04$



(a) The function e for N = M = 20.



(b) The function e for N = M = 40.

Figure 3.4: Plot of the absolute errors function of Example 3.2.1 for $p = 3$

Table 3.6: Comparison of the absolute errors of Example 3.2.1

(x, y)	DTM	Present method
$(0.2, 0.1)$	$1.17e-08$	$3.80e-07$
$(0.2, 0.4)$	$1.48e-06$	$3.80e-06$
$(0.2, 0.7)$	$4.81e-05$	$7.86e-06$
$(0.5, 0.1)$	$3.69e-06$	$5.91e-06$
$(0.5, 0.4)$	$1.28e-05$	$2.77e-05$
$(0.5, 0.7)$	$1.06e-04$	$4.81e-05$
$(0.8, 0.1)$	$7.69e-05$	$2.52e-05$
$(0.8, 0.4)$	$1.82e-04$	$1.02e-04$
$(0.8, 0.7)$	$3.90e-04$	$1.69e-04$
$(1.0, 0.9)$	$1.85e-03$	$4.12e-04$

Example 3.2.2 *Consider the following linear 2D-VIEs [1]*

$$u(x, y) = g(x, y) - \int_0^x \int_0^y \sin(x - y - t + s)u(t, s)dsdt, \qquad (x, y) \in [0, 1) \times [0, 1),$$

where $g(x, y) = (x - y + 1)\cos(x + y) - (x - y)\cos(x - y)$ *is chosen so that the exact solution is* $u(x, y) = \cos(x + y)$.

The absolute errors function for $p = 3$ *and* $N = M = 20$ *are plotted in Fig. 3.5.*

The numerical results for $p = 3$ *and* $h = k = 0.1, 0.05$ *are compared with the numerical results obtained by using two-dimensional orthogonal triangular functions (2D-TFs) [1] in Table 3.7, we observed that the results obtained by TCM are more accuracy than the results obtained by 2D-TFs.*

Table 3.7: Comparison of the absolute errors of Example 3.2.2

(x, y)	2D-TFs	Present method	
		$N = M = 10$	$N = M = 20$
$(0, 0)$	0	0	0
$(0.1, 0.1)$	$6.85e-04$	$4.41e-10$	$5.60e-10$
$(0.2, 0.2)$	$3.47e-04$	$4.96e-10$	$4.52e-10$
$(0.3, 0.3)$	$9.94e-04$	$3.22e-10$	$2.30e-10$
$(0.4, 0.4)$	$3.88e-03$	$4.18e-10$	$2.94e-10$
$(0.5, 0.5)$	$7.68e-05$	$5.30e-09$	$2.07e-09$
$(0.6, 0.6)$	$8.37e-03$	$1.86e-08$	$3.98e-09$
$(0.7, 0.7)$	$7.65e-03$	$6.07e-08$	$9.32e-09$
$(0.8, 0.8)$	$8.70e-03$	$1.65e-07$	$2.60e-08$
$(0.9, 0.9)$	$1.50e-02$	$3.90e-07$	$6.12e-08$

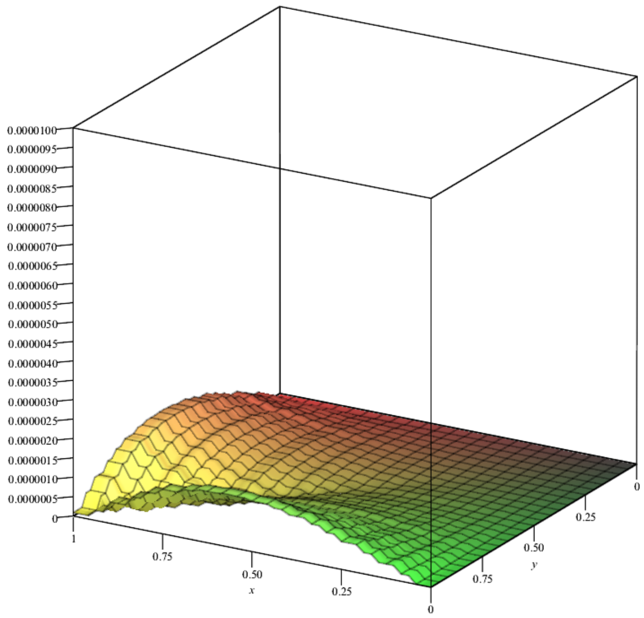


Figure 3.5: Plot of the absolute errors function of Example 3.2.2 for $p = 3$, $N = M = 20$

CONCLUSION AND PERSPECTIVE

In this dissertation, we have developed a numerical method by using Taylor collocation method for the numerical solution of two-dimensional Volterra integral equations (2.1.1). This method is easy to implement and the coefficients of the approximate solution are determined by iterative formulas without the need to solve any system of algebraic equations. Moreover, many numerical examples were introduced showing that the method is convergent with a good accuracy and the numerical results confirmed the theoretical estimates.

Further researches will be conducted by generalizing this method to approximate Volterra integral equations in three dimensional :

$$u(x, y, z) = g(x, y, z) + \int_0^x \int_0^y \int_0^z K(x, y, z, t, s, r)u(t, s, r)drdsdt.$$

BIBLIOGRAPHY

[1] E. Babolian, K. Maleknejad, M. Roodaki and H. Almasieh, Two-dimensional triangular functions and their applications to nonlinear 2D VolterraFredholm integral equations. Computers and Mathematics with Applications, 60(6) (2010), 1711-1722.

[2] J. Bongsoo, Comments on solving class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method. J. Comput. Appl. Math. 233 (2009), 224230.

[3] H. Brunner, Collocation methods for Volterra integral and related functional differential equations, Cambridge university press, Cambridge, (2004).

[4] H. Brunner, The numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation, IMA Journal of Numerical Analysis(1989) 9, 47-59.

[5] S.R. Ghorpade and B.V. Limaye, A course in multivariable calculus and analysis, Springer Science+Business Media, New York, (2010).

[6] P.C. Hansen, T.K. Jensen, Large-scale methods in image deblurring, in: Proceedings of the LNCS of Lecture Notes in Computer Science, 4699, 2007.

[7] A.V. Manzhirov, On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology, J. Appl. Math. Mech. 49 (6) (1985) 777782.

[8] M.V. Mirkin, A.J. Bard, Multidimensional integral equations. part 1. a new approach to solving microelectrode diffusion problems, J. Electroad. Chem. 323 (12) (1992) 127.

[9] B. G. Pachpatte, Integral and finite difference inequalities and applications, Netherlands elsevier, (2006).

[10] B. G. Pachpatte, Multidimensional integral equations and inequalities, Springer Science and Business Media, Vol. 9, (2011).

[11] B. G. Pachpatte, On discrete inequalities related to Gronwall's inequality, Proc. Indian Acad. Sci., Vol. 85 A, No. 1, (1977), pp 26-40.

[12] J. Radlow, A two-dimensional singular integral equation of diffraction theory, Bull. Am. Math. Soc. 70 (4) (1964) 596599.

[13] A. Tari, M.Y. Rahimi, S. Shahmorad, F. Talati, Solving a class of two-dimensional linear and nonlinear volterra integral equations by the differential transform method, J. Comput. Appl. Math. 228 (1) (2009) 7076.

[14] A. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications, Higher Education Press Beijing, New York, (2011).